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Finite-time, Event-triggered Tracking Control of Quadrotors

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Abstract In this paper, we present a novel quaternion-based event triggered control strategy for trajectory tracking with a quadrotor that is suitable for implementation on digital platforms with hardware constraints. The proposed control ensures asymptotic convergence to a desired position trajectory and finite time convergence to a desired attitude trajectory. We also present Lyapunov based analysis to demonstrate validity of the triggering scheme and also rule out Zeno behaviour. The performance of the event triggered control laws are demonstrated through numerical simulations.

1 Introduction

With advances in autonomous quadrotor technology and increased use in various industries from entertainment to defence, it seems prudent to develop feedback control schemes felicitous to implementation on digital platforms such as microcontrollers or computers.

Traditional continuous control schemes for quadrotors have been extensively explored in [1], [2], [3], [4], [5] among other articles by utilizing various linear and nonlinear control methods. [1] explores Proportional Integral Derivative controllers for quadrotors, [2] delves into feedback linearization and sliding mode based control design, [3] discusses robust approaches, [4] proposes neural networks as the control paradigm while [5] employs adaptive nonlinear control for fault tolerant operation.

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All of the above eclectic mix of methods aim to achieve asymptotic convergence of position states which is difficult to prove theoretically owing to the asymptotic nature of both the inner attitude and outer position controller. This has lead to research in finite-time convergence schemes as in [6], [7], [8] and [9]. In [6] and [7], geometric methods are used to obtain finite-time stable control inputs for position tracking in some aerial vehicles. In [8] and [9], finite time attitude stabilisation of spacecraft is explored.

Although the above finite-time schemes guarantee position tracking in theory, they demand continuous in time control inputs, which are in reality implemented only periodically on digital platforms. This approach is exceedingly hardware dependent and at times, the hardware may not be able to accommodate a small enough sampling period for the practically implemented discrete control inputs to behave like their theoretical continuous counterparts. This prompts development of event-triggered control schemes (where control is updated at discrete points in time when a state-dependent trigger condition is satisfied as detailed in [14]); such as those explored in [10] and [11]. In [10] and [11], an event-triggered nonlinear scheme is presented for attitude stabilisation of a quadrotor using an event-triggered version of the Sontag's universal controller. This results in a rather complex controller for the quadrotor attitude and also doesn't have finite-time convergence properties.

In this article, we address, firstly, both position and attitude controllers for the quadrotor. While the outer-loop position controller is asymptotic, the inner-loop attitude controller is finite-time, thus guaranteeing position tracking in theory though the control design is modular. Event-triggered versions of both laws are developed to enhance practical implementability on digital platforms.

2 System Model

In this section, we introduce the translational and unit quaternion-based rotational dynamics of a quadrotor system.

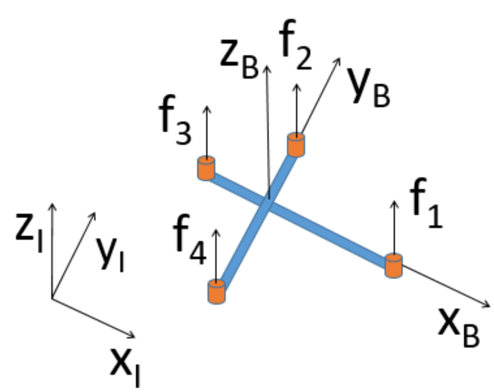

Fig. 1 Quadrotor - Reference Frames: body (B) frame and inertial (I) frame

2.1 Translational Dynamics

We denote the position vector by $\mu = (x,y,z)^T \in \mathbb{R}^3$ with dynamics,

$$\ddot{\mu} = \begin{bmatrix}0 & 0 & -9.8\end{bmatrix}^T + \mathbf{R}e_3\frac{u_1}{m}, \quad \mu(0) = \mu_0 \tag{1}$$

where $e_3 = (0,0,1)^T$, $m \in \mathbb{R}^+$ is the mass of the quadrotor and $\mathbf{R} \in SO(3)$ is the 3x3 rotation matrix from the quadrotor body frame to the inertial frame. The feedback $u_1(\cdot) \in \mathbb{R}$ is the sum of thrust forces from the individual motors.

2.2 Rotational Dynamics

Consider unit quaternion $(q_0, q_v)^T \in \mathbb{R}^4$ representing the orientation of the quadrotor in body frame with respect to the inertial frame. Their dynamics are given as:

$$\begin{aligned}\dot{q_0} &= -\frac{1}{2}q_v^T\omega, \quad q_0(0) = \bar{q}_0\\ \dot{q_v} &= \frac{1}{2}(q_0\omega + q_v \times \omega), \quad (q_0^2 + q_V^T q_v = 1), \quad q_v(0) = \bar{q}_v\end{aligned} \tag{2}$$

Further, from the Euler's equations of motion, we have

$$J\dot{\omega} = J\omega \times \omega + \tau, \quad \omega(0) = \bar{\omega} \tag{3}$$

where $J \in \mathbb{R}^{3\times 3}$ is the symmetric, positive-definite moment of inertia matrix of the quadrotor, $\omega \in \mathbb{R}^3$ is the angular velocity of the qaudrotor in the body (B) frame and $\tau = (\tau_{x_B}, \tau_{y_B}, \tau_{z_B})^T \in \mathbb{R}^3$ is the torque provided in the body frame through differential thrust forces of individual motors.
The control for the translation dynamics u_1 and rotation dynamics $\tau = (\tau_{x_B}, \tau_{y_B}, \tau_{z_B})^T$ can be expressed in terms of the angular velocities of the individual motors ω_{m1}, ω_{m2}, ω_{m3}, ω_{m4} as follows,

$$\begin{bmatrix}u_1\\ \tau_{x_B}\\ \tau_{y_B}\\ \tau_{z_B}\end{bmatrix} = \begin{bmatrix}f_1+f_2+f_3+f_4\\ d(f_2-f_4)\\ d(f_3-f_1)\\ \frac{c_Q}{c_T}(-f_1+f_2-f_3+f_4)\end{bmatrix} = \begin{bmatrix}c_T & c_T & c_T & c_T\\ 0 & dc_T & 0 & -dc_T\\ -dc_T & 0 & dc_T & 0\\ -c_Q & c_Q & -c_Q & c_Q\end{bmatrix}\begin{bmatrix}\omega_{m1}^2\\ \omega_{m2}^2\\ \omega_{m3}^2\\ \omega_{m4}^2\end{bmatrix} \tag{4}$$

with all motors being assumed identical and producing thrust f_i. Further, $c_T > 0$ is the coefficient of thrust of each motor, $c_Q > 0$ is the coefficient of torque of each motor and $d > 0$ is the distance between the centre of mass of the quadrotor and the centre of a motor (half-arm distance). Equation (4) has been derived in [1] and [5].

2.3 Preliminaries and control objective

Given a reference position trajectory $\mu_r \in \mathbb{R}^3$ with initial value $\bar{\mu}_r$, we define translation errors $e_1, e_2 \in \mathbb{R}^3$ as,

$$\begin{aligned} e_1 &:= \mu - \mu_r \\ e_2 &:= \dot{\mu} - \dot{\mu}_r + k_1 e_1, \qquad k_1 > 0 \end{aligned} \tag{5}$$

We also define an error quaternion $s \in \mathbb{R}^4$ (given reference quaternion $q_r \in \mathbb{R}^4$ with initial value $\bar{q}_r$) as follows:

$$s := \begin{bmatrix} s_0 \ s_v \end{bmatrix}^T = q_r^{-1} * q \tag{6}$$

where $*$ represents quaternion multiplication.The reader is refeerd to [16, section 3.5] for further details on quaternion representation of rotation. Computation of the corresponding error rotation matrix $R(s) \in SO(3)$ and error in angular velocity $\delta\omega \in \mathbb{R}^3$ (given reference angular velocity $\omega_r \in \mathbb{R}^3$ with initial value $\bar{\omega}_r$) are shown below.

$$R(s) = R(q_r^{-1})R(q) = R(q_r)^T R(q) \tag{7}$$

$$\delta\omega = \omega - R(s)\omega_r \tag{8}$$

The following result is crucial for subsequent Lyapunov analysis of the finite-time attitude control strategy.

Lemma 1. *Consider non-negative numbers α and β and an integer $p_1 > 1$, then we have (due to concavity of function $f(z) = z^{1/p_1}$)*

$$\alpha^{1/p_1} + \beta^{1/p_1} \geq (\alpha+\beta)^{1/p_1} \tag{9}$$

For the above setup we are now ready to state our control objective.

Control Objective: To design an event triggered feedback strategy (u, τ) to ensure that the origin $(e_1, e_2) = (\bar{0}_{3\times1}, \bar{0}_{3\times1})$ defined in (5) for dynamics (1) is asymptotically stable.

3 Continuous Feedback Control

3.1 Position Control

The translational error dynamics can be written as follows:

$$\begin{aligned} \dot{e}_1 &= e_2 - k_1 e_1 \\ \dot{e}_2 &= \begin{bmatrix} 0 \ 0 \ -9.8 \end{bmatrix}^T + \mathbf{R} e_3 \frac{u_1}{m} + k_1 \dot{e}_1 \end{aligned} \tag{10}$$

Theorem 1. *Consider the translational dynamics given in* (10) *and define* $\sigma \in \mathbb{R}^3$ *as*

$$\sigma = \begin{bmatrix} 0 \; 0 \; 9.8 \end{bmatrix}^T - k_1\dot{e}_1 - e_1 - k_2 e_2, \quad k_1 > 0, k_2 > 0 \tag{11}$$

where e_1 *and* e_2 *are as defined in* (5)*. Then, the feedback control* $u_1 \in \mathbb{R}^3$ *solving,*

$$\mathbf{R}.e_3 \frac{u_1}{m} = \sigma \tag{12}$$

with m being the mass of the quadrotor and $\mathbf{R}$ *being the rotation matrix corresponding to the quadrotor's body frame, will exponentially stabilize the origin of the translational error dynamics given in* (10).

Proof. Consider the Lyapunov function and its derivative as below,

$$\begin{aligned} V &= \frac{1}{2}e_1^2 + \frac{1}{2}e_2^2 \\ \dot{V} &= e_1^T\dot{e_1} + e_2^T\dot{e_2} \\ &= e_1^T(e_2 - k_1 e_1) + e_2^T(\begin{bmatrix} 0 \; 0 \; -9.8 \end{bmatrix}^T + \mathbf{R}.e_3\frac{u_1}{m} + k_1\dot{e_1}) \end{aligned} \tag{13}$$

Upon substituting for $\mathbf{R}.e_3\frac{u_1}{m}$ from (12) in (13), we obtain

$$\dot{V} = -k_1\|e_1\|^2 - k_2\|e_2\|^2 < 0 \tag{14}$$

The negative definiteness of V and the fact that $\dot{V} \leq -\gamma V$ for some $\gamma > 0$, implies global exponential stability of the origin of (10). □

The control law proposed in Theorem 1 satisfies upon multiplication with its transpose,

$$e_3^T R^T R e_3 (u_1/m)^2 = \|\sigma\|^2 \implies (u_1/m)^2 = \|\sigma\|^2 \implies u_1 = m\|\sigma\| \tag{15}$$

Remark. Based on the control law (12) in Theorem 1, the desired rotation matrix ($\mathbf{R}$) corresponding to the reference quaternion q_r ($\mathbf{R} := \mathbf{R}(q_r)$) can be obtained after substituting (15) in (12) as follows

$$\mathbf{R}(q_r)e_3\|\sigma\| = \sigma \implies \mathbf{R}(q_r)e_3 = \sigma/\|\sigma\| = \hat{\sigma} \quad \text{(a unit vector)} \tag{16}$$

Upon closer inspection, equation (16) can be interpreted as defining a rotation matrix that rotates the inertial z-axis, e_3 to the body z-axis, $\hat{\sigma}$. Suppose we have, $e_3.\hat{\sigma} = cos(\phi)$ and $e_3 \times \hat{\sigma} = e_v$, then a choice of quaternion that accomplishes this rotation is

$$q_r = \begin{bmatrix} cos(\phi/2) \; {e_v}^T sin(\phi/2) \end{bmatrix}^T \tag{17}$$

Theorem 1 is based on a similar formulation in [7].

3.2 Attitude Control

The error quaternion dynamics derived from (2)-(3) are as follows

$$\dot{s_0} = -\frac{1}{2}s_v^T\delta\omega$$
$$\dot{s_v} = \frac{1}{2}(s_0\delta\omega + s_v \times \delta\omega) \tag{18}$$

where $\delta\omega = \omega - \mathbf{R}(s)\omega_r$ is the error angular velocity. The error angular velocity dynamics is obtained from equation (3):

$$J\dot{\delta\omega} = J\omega \times \omega + \tau - J\Phi \tag{19}$$

where

$$\Phi = \frac{d(R(s)\omega_r)}{dt} \tag{20}$$

Equations (18) and (19) have been derived in detail in [15].

Theorem 2. *Consider the rotational error dynamics given by* (18) *and* (19)*. Define* $\delta\omega_d \in \mathbb{R}^3$ *as,*

$$\delta\omega_d := \frac{-k_p s_v}{(s_v^T s_v)^{1-\frac{1}{p_1}}}, \qquad p_1 \in (1,2) \quad and \quad k_p > 0 \tag{21}$$

Then the feedback control $\tau \in \mathbb{R}^3$ *given by,*

$$\tau = -s_v - J\omega \times \omega + J\Phi + J\dot{\delta\omega}_d - \frac{0.5Je}{(0.5e^TJe)^{1-1/p_1}}\frac{k_p}{2^{1/p_1}} \tag{22}$$

where

$$e = \delta\omega - \delta\omega_d \in \mathbb{R}^3 \tag{23}$$

stabilizes the $s_0 = 1, s_v = \bar{0}_{3\times 1}, \delta\omega = \bar{0}_{3\times 1}$ *equilibrium of the rotational error dynamics in finite time.*

Theorem 2 is based on a similar formulation for finite time convergence to desired attitude as outlined in [6].

Proof. From (19) and (23), we have the following attitude error dynamics:

$$J\dot{e} = J\omega \times \omega + \tau - J\Phi - J\dot{\delta\omega}_d \tag{24}$$

Consider a Lyapunov candidate and its directional derivative along (18) and (24) as below.

$$\begin{aligned}
V_2 &= (1-s_0)^2 + s_v^T s_v + \frac{1}{2}e^T J e = 2(1-s_0) + \frac{1}{2}e^T J e > 0 \\
\dot{V}_2 &= s_v^T \delta\omega + e^T J \dot{e} \\
&= s_V^T (e + \delta\omega_d) + e^T (J\omega \times \omega + \tau - J\Phi - J\delta\dot{\omega}_d) \\
&= -k_p (s_v^T s_v)^{1/p_1} + e^T (s_v + J\omega \times \omega + \tau - J\Phi - J\delta\dot{\omega}_d)
\end{aligned} \tag{25}$$

where the final equation is arrived at by substituting for $\delta\omega_d$ from (21). Further using (22) in (25),

$$\begin{aligned}
\dot{V}_2 &= -k_p (s_v^T s_v)^{1/p_1} - \frac{k_p}{2^{1/p_1}} (0.5 e^T J e)^{1/p_1} \\
&\leq -\frac{k_p}{2^{1/p_1}} (V_1^{1/p_1} + (0.5 e^T J e)^{1/p_1}) \\
&\leq -\frac{k_p}{2^{1/p_1}} (V_2^{1/p_1})
\end{aligned} \tag{26}$$

The transition from the first inequality to the second in (26) can be explained by a direct application of Lemma 1. The above inequality guarantees finite time convergence of the quaternion and angular velocity errors to zero. □

Theorem 3. *Consider the rotational and translational error dynamics given by* (10)*,* (18) *and* (19)*. Then the feedback control given by* (12) *and* (22) *with q_r defined by* (16)-(17) *and ω_r obtained from kinematics of q_r identical to* (2)*, guarantees asymptotic convergence of (e_1, e_2) to $(\bar{0}_{3\times1}, \bar{0}_{3\times1})$.*

Proof. This is a direct consequence of finite-time convergence of the attitude error using Theorem 2 employed in the controller of Theorem 1. □

4 Event-triggered Control

In this section, we extend the continuous time control formulations in Section 3 to event-triggered strategies by using triggering functions as detailed below. We also validate the proposed triggering functions by ruling out Zeno behaviour.

Theorem 4. *Consider the translational dynamics given in* (10)*. Define a triggering function $f_1 : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+ \to \mathbb{R}$,*

$$f_1(e_1, e_2, t) = -\dot{V} + \mu(-k_1 \|e_1\|^2 - k_2 \|e_2\|^2), \quad 0 < \mu < 1 \tag{27}$$

with V defined in (13) *and an event-triggered $\sigma(t)$ as follows:*

$$\begin{cases} \sigma(t) = \begin{bmatrix} 0 \ 0 \ 9.8 \end{bmatrix}^T - k_1 \dot{e}_1 - e_1 - k_2 e_2, \quad if \quad f_1(e_1, e_2, t) < 0 \\ \dot{\sigma}(t) = \bar{0}_{3\times1}, \quad otherwise \end{cases} \tag{28}$$

for arbitrary $k_1 > 0, k_2 > 0$. *Then, with feedback control solving* (12) *for the modified* σ *above, we have exponential stability of the origin of the translational error dynamics given in* (10) *and no Zeno behaviour is observed.*

Remark: The modified σ ensures that there is no change in feedback control derived using $\sigma(t)$ unless the triggering condition in (28) is satisfied.

Proof. While the trigger condition $f_1(e_1,e_2,t) \leq 0$ in (28) is satisfied, we have $\sigma(t) = \left[0\ 0\ 9.8\right]^T - k_1\dot{e}_1 - e_1 - k_2e_2, \quad k_1 > 0, k_2 > 0$ and hence, from (13)-(14), we have exponential stability of the origin for translational errors given in (10). Further, when $f_1(e_1,e_2,t) > 0$, we have directly from (27), $\dot{V} < \mu(-k_1\|e_1\|^2 - k_2\|e_2\|^2) < 0$ and hence exponential stability of origin of translational errors using the common Lyapunov function method for switched systems in (10) is obtained.

To rule out Zeno behaviour, it is sufficient to show that the inter-execution time between two events has a positive lower bound (Minimum sampling interval). Let us consider t_1 and t_2 to represent times when consecutive events are triggered.

$$\begin{aligned} for \quad t_1 < t < t_2, \qquad V &= \frac{1}{2}e_1^2 + \frac{1}{2}e_2^2 \\ \dot{V} &< \mu(-k_1\|e_1\|^2 - k_2\|e_2\|^2) < 0 \\ \text{also, } \dot{e}_1 = e_2 - k_1e_1 \quad &\text{and} \quad \dot{e}_2 = -e_1 - k_2e_2 \end{aligned} \tag{29}$$

Further, since $V(e_1,e_2) > 0$ and $\dot{V}(e_1,e_2) < 0$, we have boundedness of $V(e_1,e_2)$ which in turn implies that e_1 and e_2 are bounded. Let G_1 and G_2 denote their upper bounds respectively. We obtain norm of the derivative of the error vector as follows:

$$\begin{aligned} ||\dot{E}_t|| := \sqrt{\dot{e}_1^T\dot{e}_1 + \dot{e}_2^T\dot{e}_2} &\leq \sqrt{(e_2 - k_1e_1)^T(e_2 - k_1e_1) + (-e_1 - k_2e_2)^T(-e_1 - k_2e_2)} \\ &\leq \sqrt{|||e_1|||^2(1+k_1^2) + |||e_2|||^2(1+k_2^2) + 2e_1^Te_2(k_2 - k_1)} \\ ||\dot{E}_t|| &\leq \sqrt{G_1^2(1+k_1^2) + G_2^2(1+k_2^2) + 2G_1G_2(k_2 - k_1)} := \alpha_p \end{aligned} \tag{30}$$

Upon integration of (30), we have

$$\int_{t_1}^{t_2} ||\dot{E}_t||\mathrm{d}t \leq \alpha_p(t_2 - t_1) \tag{31}$$

We also have

$$\int_{t_1}^{t_2} ||\dot{E}_t||\mathrm{d}t \geq ||\int_{t_1}^{t_2} \dot{E}_t\mathrm{d}t|| = ||E_t(t_2) - E_t(t_1)|| = D \tag{32}$$

where D is a finite constant that can be computed from the bounds G_1 and G_2. Though the analysis holds in a strictly open interior of $[t_1, t_2]$ this can be extended by continuity to $[t_1, t_2]$. Therefore we have,

$$D \leq \alpha_p(t_2 - t_1) \implies (t_2 - t_1) \geq D/\alpha_p \tag{33}$$

□

Theorem 5. *Consider the rotational error dynamics given in* (18) *and* (19) *and definitions in* (20)*,* (21) *and* (23)*. Define a triggering function* $f_2 : \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+ \to \mathbb{R}$

$$f_2(s_0, s_v, \delta\omega, t) = -\dot{V}_2 + \frac{-k_p}{2^{1/p1}} V_2^{1/p1} \tag{34}$$

with V_2 *defined in* (25) *and an event-triggered feedback control* $\tau(t)$ *as follows*

$$\begin{cases} \tau(t) = -s_v - J\omega \times \omega + J\Phi + J\delta\dot{\omega}_d - \frac{0.5Je}{(0.5e^TJe)^{1-1/p_1}} \frac{k_p}{2^{1/p_1}}, & if \quad f_2(s_0, s_v, \delta\omega, t) < 0 \\ \dot{\tau}(t) = \bar{0}_{3\times 1}, \quad otherwise \end{cases} \tag{35}$$

for some positive constant k_p*. Then the* $s_0 = 1$, $s_v = \bar{0}_{3\times 1}, \delta\omega = \bar{0}_{3\times 1}$ *equilibrium of the rotational error dynamics given in* (18)–(19) *is stabilized in finite time and no Zeno behaviour is observed.*

Proof. While the trigger condition $f_2(s_0, s_v, \delta\omega, t) \leq 0$ is satisfied we have, $\tau(t) = -s_v - J\omega \times \omega + J\Phi + J\delta\dot{\omega}_d - \frac{0.5Je}{(0.5e^TJe)^{1-1/p_1}} \frac{k_p}{2^{1/p_1}}$ and hence, from (25)-(26), we have finite time stability of the origin of rotational errors given in (18) and (19). Also, when $f_2(s_0, s_v, \delta\omega, t) > 0$, we have directly from (34), $\dot{V}_2 < \frac{-k_p}{2^{1/p1}} V_2^{1/p1}$ and hence finite time stability of origin of rotational errors given in (18) and (19).

To analyse Zeno behavior, we consider as before, t_a and t_b, the consecutive trigger instants.

$$\begin{aligned} for \quad t_a < t < t_b, \qquad V_2 &= (1 - s_0)^2 + s_v^T s_v + \frac{1}{2}(\delta\omega - \delta\omega_d)^T J(\delta\omega - \delta\omega_d) > 0 \\ \dot{V}_2 &< \frac{-k_p}{2^{1/p1}} V_2^{1/p1} < 0 \\ \text{also, } \dot{s_0} &= -\frac{1}{2} s_v^T \delta\omega \quad \text{and} \quad \dot{s_v} = \frac{1}{2}(s_0\delta\omega + s_v \times \delta\omega) \\ \text{and, } \dot{\delta\omega} - \delta\dot{\omega}_d &= -J^{-1}s_v - \frac{0.5e}{(0.5e^TJe)^{1-1/p_1}} \frac{k_p}{2^{1/p_1}} \end{aligned} \tag{36}$$

Further, since $V_2(s_0, s_v, \delta\omega) > 0$ and $\dot{V}_2(s_0, s_v, \delta\omega) < 0$, we have boundedness of $V_2(s_0, s_v, \delta\omega)$ which in turn implies that s_0, s_v and $\delta\omega$ (and hence also e = $\delta\omega - \delta\omega_d$) are bounded. Let G_0, G_v and G_ω (and also G_e) denote their upper bounds respectively. We obtain the norm of the derivative of the combined error vector as follows:

$$
\begin{aligned}
||\dot{E}_r|| &= \sqrt{\dot{s}_0^T\dot{s}_0 + \dot{s}_v^T\dot{s}_v + \dot{e}^T\dot{e}} \\
&\leq \Bigg\{ \frac{1}{4}(s_v^T\delta\omega\delta\omega^T s_v) + \frac{1}{4}(s_0^2\delta\omega^T\delta\omega + (s_v\times\delta_\omega)^T(s_v\times\delta_\omega) + 2s_0\delta\omega^T(s_v\times\delta_\omega)) \\
&\quad + s_v^T s_v + \frac{s_v^T J e}{(0.5e^TJe)^{1-1/p_1}}\frac{k_p}{2^{1/p_1}} + \frac{1}{4}\frac{e^TJ^TJe}{(0.5e^TJe)^{2-2/p_1}}\frac{k_p^2}{2^{2/p_1}} \Bigg\}^{1/2} \\
||\dot{E}_r|| &\leq \Bigg\{ \frac{1}{4}(G_v^2G_\omega^2) + \frac{1}{4}(G_0^2G_\omega^2 + G_v^2G_\omega^2 + 0) \\
&\quad + G_v^2 + \frac{JG_vG_e}{(0.5JG_e^2)^{1-1/p_1}}\frac{k_p}{2^{1/p_1}} + \frac{1}{4}\frac{J^TJG_e^2}{(0.5JG_e^2)^{2-2/p_1}}\frac{k_p^2}{2^{2/p_1}} \Bigg\}^{1/2} := \alpha_r
\end{aligned}
\tag{37}
$$

Upon integration of (37), we have

$$
\int_{t_a}^{t_b} ||\dot{E}_r|| \mathrm{d}t \leq \alpha_r(t_b - t_a) \tag{38}
$$

We also have,

$$
\int_{t_a}^{t_b} ||\dot{E}_r|| \mathrm{d}t \geq || \int_{t_a}^{t_b} \dot{E}_r \mathrm{d}t|| = ||E_r(t_2) - E_r(t_1)|| = B \tag{39}
$$

where B is a constant that can be related directly to G_0, G_v and G_ω and also G_e. Therefore we have,

$$
B \leq \alpha_r(t_a - t_b) \implies (t_a - t_b) \geq B/\alpha_r \tag{40}
$$

The above analysis shows that the time between consecutive events $(t_b - t_a)$ is lower bounded by B/α_r for the attitude event triggering scheme, thus precluding Zeno behavior. □

Theorem 6. *Consider the rotational and translational error dynamics given by* (10), (18) *and* (19). *Then the event-triggered feedback law given by* (12), (28) *and* (35) *with q_r defined by* (16)-(17) *and ω_r obtained from kinematics of q_r identical to* (2) *guarantees asymptotic convergence of (e_1, e_2) to $(\bar{0}_{3\times1}, \bar{0}_{3\times1})$.*

The proof of the above result follows along the same logic as Theorem 3.

5 Results

Figures 2, 3 and 4 depict the working of the event-triggered exponential and finite time control strategies. They were obtained via numerical simulations using MATLAB©'s ODE solvers. Mass and inertia were taken as 1.79 kg and $0.03I_{3\times3}$ which correspond to the QBall 2© quadrotor as detailed in [5]. Further, gains

k_1, k_2, k_p and p_1 were chosen as $2.1, 2.1, 0.0009$ and 1.6 respectively. Initial conditions on position were taken as $\begin{bmatrix}0 & 1 & 0\end{bmatrix}$ and on body quaternions were taken as $q_0 = 1$ and $q_v = \begin{bmatrix}0 & 0 & 0\end{bmatrix}$. Fig. 2 shows a three dimensional visualization of the quadrotor (depicted by the actual trajectory) tracking the reference trajectory. Fig. 3 depicts the time evolution of the 4 components of the actual (quadrotor's) quaternion against the reference quaternion. Lastly, Fig. 4, plots the triggering instants for the poistion and attitude controllers. A $\circ$ represents that an event was triggered and not otherwise. The minimum inter-event time observed in these simulations was 0.0864 s. It was also observed that the attitude controller requires faster control changes in the beginning (up to about $8s$) which approximately coincides with the attitudes coming close to the desired values beyond which only minor corrections are required hence sparse actuation beyond. The position control continues to trigger at slower rates than the attitude throughout. Further, upon addition of a sinusoidal disturbance of low frequency (20 Hz) and amplitude (10% maximum value), the proposed control strategy continues to perform accurately. Figures in 5 depict position and quaternion convergence in the presence of aforementioned disturbance. They are indistinguishable from figures 2, 3 where no disturbance was present. Only the triggering schedule differs as seen in figure 6

6 Conclusions and Future Work

An event-triggered algorithm for the quadrotor trajectory tracking problem was proposed in this work. The algorithm in theory guarantees finite time attitude convergence and exponential position trajectory tracking. From the numerical simulation results and proofs detailed in the previous sections, we can conclude the validity and efficacy of the proposed event triggered schemes. Further, testing of the schemes on hardware (Qball 2© quadrotor) with VICON© based feedback is currently underway. A facet of this work that deserves to be explored in greater detail is finding an optimal triggering function to ensure maximum inter-triggering time and also quantifying precisely the lower bound on inter-trigger times.

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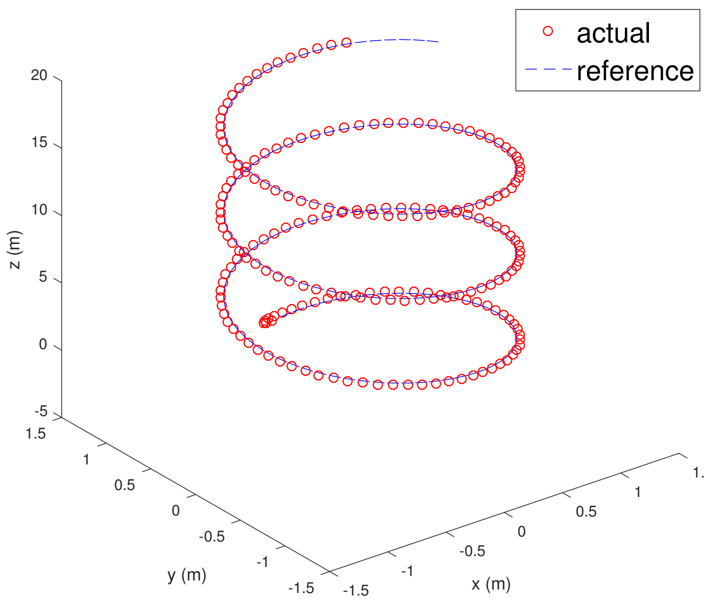

Fig. 2 3D plot depicting convergence of actual trajectory to reference trajectory under event triggered scheme

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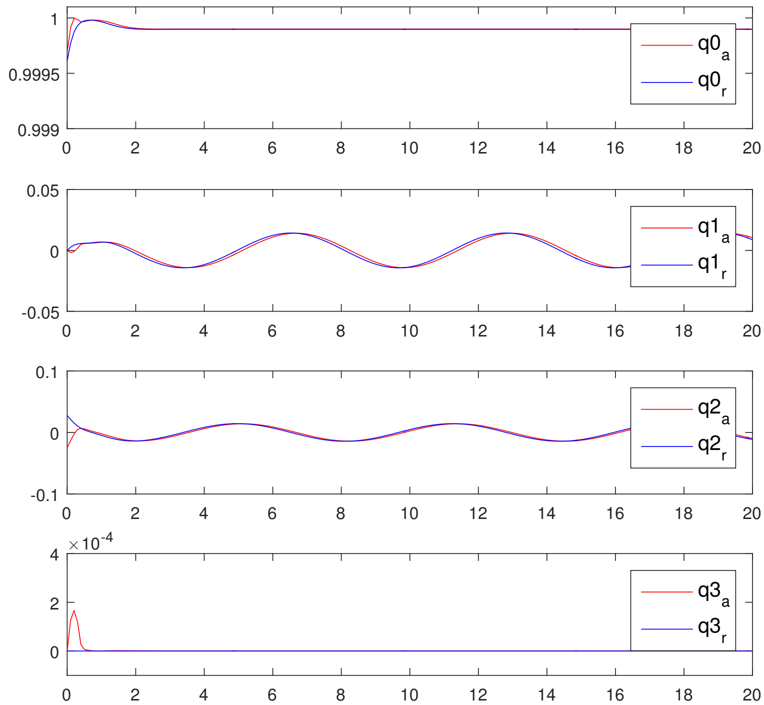

Fig. 3 Event triggered convergence of actual quaternion to reference quaternion in finite time

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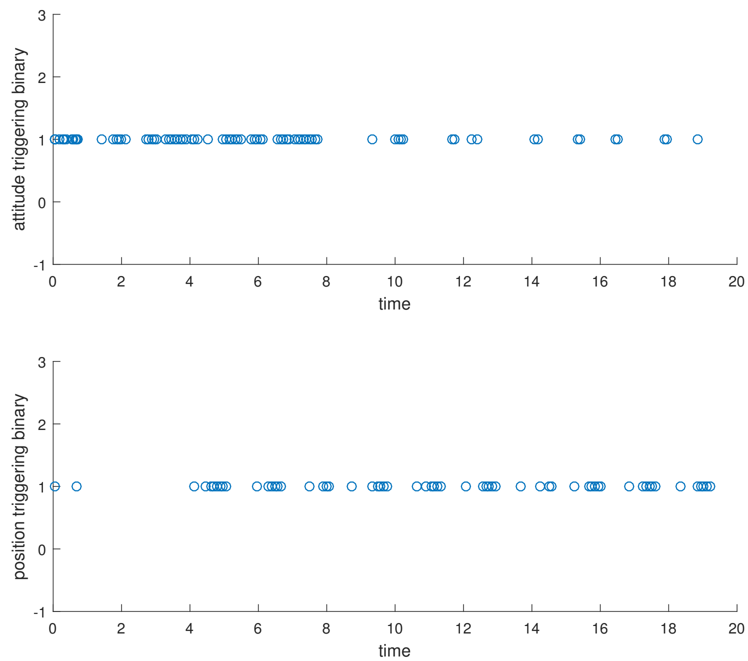

Fig. 4 Triggering binary vs time for attitude (top) and position (bottom) event triggering schemes (○ - event was triggered, otherwise - no event triggered)

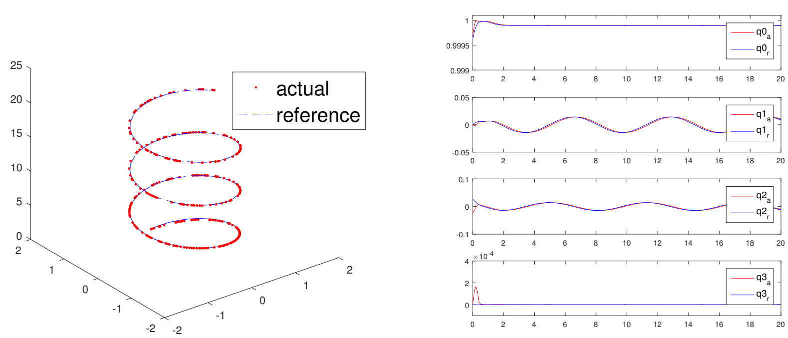

Fig. 5 Disturbance enduring convergence of trajectory (left) and quaternion (right) under event triggered scheme

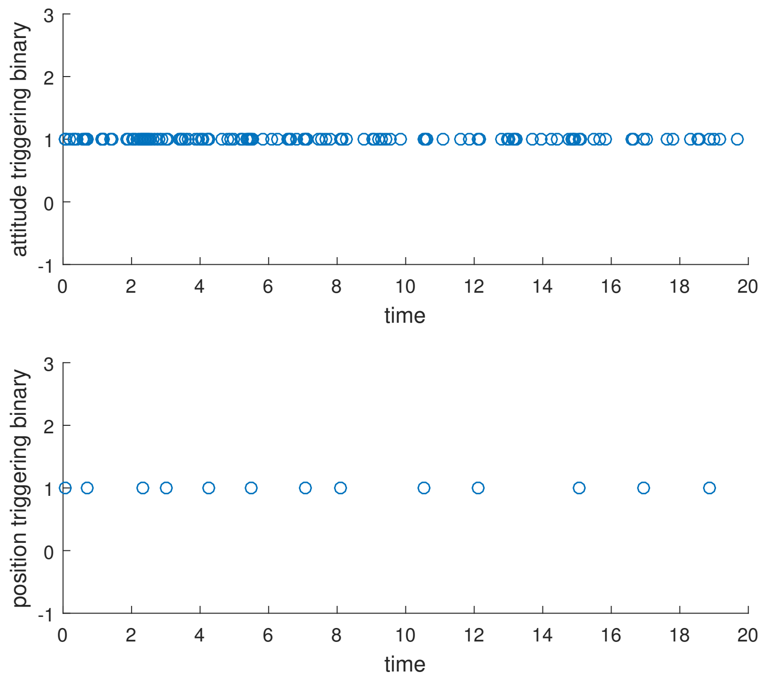

Fig. 6 Triggering schedule in the presence of disturbance