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Minimum-Time Control of Linear Systems between Arbitrary States

Fabio Curti and Marcello Romano

Abstract For the first time to the best knowledge of the authors, a solution method is here presented for the problem of minimum-time control of a general linear time-invariant normal system evolving from an arbitrary initial state to an arbitrary desired final state subjected to a cubic-constrained control. In particular it is demonstrated that the above problem can be solved by exploiting the solution of an associated minimum-time control problem from an initial state related to the boundary states of the original problem to the state-space origin. Furthermore, new analytical solutions are presented for this optimal control problem in the important case of a double integrator system. In particular, the final time and the open loop control sequences are given explicitly as a function of the boundary states and a feedback optimal control synthesis is given. Notably, exact minimum-time control solution is currently known only for the case of minimum-time control of a double integrator from an arbitrary state to the state-space origin.

1 Introduction

The theory of minimum-time control of linear systems is for the most part maturely established. In particular, general theorems regarding necessary conditions of optimality, uniqueness, non-singularity and upper-bound on the number of switchings, exist for the minimum-time control of linear systems for the case of evolution from an arbitrary initial state to the origin [14, 115–187] [12, 127–158][1, 395–426][3,

Fabio Curti
School of Aerospace Engineering, Sapienza University of Rome, Via Salaria, 851, 00138 Rome. Italy. e-mail: fabio.curti@uniroma1.it

Marcello Romano
Dept. of Mechanical & Aerospace Engineering, Naval Postgraduate School, 700 Dyer Rd., Monterey, California 93940. U.S.A. e-mail: mromano@nps.edu

Both authors have contributed equally to this work. Both authors are corresponding authors.

83–173][11, 248–249], as well as for the case of evolution from an arbitrary initial state to a compact convex target set. [12, 127–158]

However, an important knowledge gap remains regarding the minimum-time control of linear system evolving from an arbitrary initial state to an arbitrary desired final state. The main rationale of the research effort whose results are reported in the present paper was to contribute to fill this knowledge gap.

Among linear systems, the double integrator and the harmonic oscillator are widely studied as they constitute an applicable models for many dynamic phenomena encountered in engineering and science. [15][8][20][21]

Well known solutions exist for the minimum-time control –from an arbitrary initial state to the origin– of a double integrator [14, 22-35][1, 507–514][12, 40–48][4, 112–113][3, 24–30][18, 103–109][13, 211–239],[10, 184–200][11, 249–259][9, 23–74], of an undamped harmonic oscillator [5][14, 22-35][2, 323–327][1, 568–589][12, 40–48][18, 103–109][9, 23–74] and of a damped harmonic oscillator. [5][1, 590–595][12, 40–48][3, 83–173]

Furthermore, solutions exist for the minimum-time control of a double integrator from an arbitrary initial state to a target set consisting of the ordinate axis of the phase plane [14, 50–53], or to a target set consisting of the abscissa axis of the phase plane or of a segment of it containing the origin [1, 518–522] or to a point of the abscissa. [17, 218–230] A graphical outline of the minimum-time control of a double integrator from an arbitrary initial state to an arbitrary desired final state is reported in [9, 28–31]. Finally, solutions exist for the minimum-time control of a harmonic oscillator from an arbitrary initial state to a target set consisting of a circle around the origin, i.e. an oscillation at constant amplitude and energy. [14, 53–58][19]

In this paper, the following three main contributions are reported which are new and original to the best knowledge of the authors:

1. A solution method is presented for the problem of minimum-time control of a general linear time-invariant normal system evolving from an arbitrary initial state to an arbitrary desired final state subjected to a cubic-constrained control. In particular it is demonstrated that the above problem can be solved by exploiting the solution of an associated minimum-time control problem from an initial state related to the boundary states of the original problem to the state-space origin.
2. By following the method in item 1, the complete analytical solution is found and presented for the case of minimum-time control of a double integrator system between arbitrary states, including a closed-form solution of the optimal final time as a function of the initial and final states, and the optimal control history

The paper is organized as follows. Section 2 summarizes the known results valid for the minimum-time control of a general linear time-invariant normal system from an arbitrary initial state to the state-space origin. Section 3 introduces and solves the minimum-time control problem between two states related by a suitable offset. Section 4 introduces the problem statement and report the demonstrated solution method for the problem of minimum-time control between two arbitrary states. Section 5 reports the solved problem of minimum-time control of the double integrator between arbitrary states. Finally, section 7 concludes the paper.

2 Minimum-time control of a linear system from an arbitrary state to the state-space origin

This section contains the problem statement and summarizes known facts regarding the minimum-time control of a linear time-invariant normal system from an arbitrary initial state to the origin of the state space. [14, 115–187] [12, 127–158] [1, 395–426] [3, 83–173][11, 248–249]

Problem 1. ***(Minimum-time control of a linear system from an arbitrary state to the state-space origin)***
Given a Linear Time Invariant normal dynamic system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{1}$$

where
$\mathbf{x}(t) \in \mathscr{R}^n$ is the state vector,
$\mathbf{A} \in \mathscr{R}^{n\times n}$ is the matrix of the dynamics,
$\mathbf{B} \in \mathscr{R}^{n\times r}$ is the distribution matrix of the control,
$\mathbf{u}(t) \in \mathscr{R}^r$ is the control vector.

Assume that the system is normal [1, th.6-6, p.400] and that the components of the control vector are cubic-constrained in magnitude as follows

$$|u_j(t)| \leq 1 \quad j = 1, 2, ..., r. \tag{2}$$

Given an arbitrary initial state, at the initial time $t = t_0 = 0$,

$$\mathbf{x}(0) = \xi, \tag{3}$$

find the control $\mathbf{u}^*(t)$ that transfers the system in a minimum time T^* to the state-space origin, i.e. such that

$$\mathbf{x}(T^*) = \mathbf{0}. \tag{4}$$

2.1 Necessary Conditions for the Solution of Problem 1

According to the minimum principle of Pontryagin, the optimal state trajectory $\mathbf{x}^*(t)$, costate trajectory $\mathbf{p}^*(t) \in \mathscr{R}^n$ and control history $\mathbf{u}^*(t)$ satisfy the following necessary conditions of optimality [1, th.6-4, p.396], and in particular it yields

$$\mathbf{u}^*(t) = -sgn\left(\mathbf{B}'\mathbf{p}^*(t)\right). \tag{5}$$

2.2 Properties of the solution of Problem 1

Existence of the solution: if the state matrix $\mathbf{A}$ has eigenvalues with non-positive real parts, a time-optimal control $\mathbf{u}^*(t)$ exists which transfers any initial state $\mathbf{x}(t_0)$ to the origin of the state space [1, th.6-10, p.420].

Since the system of Eq. 1 is normal by assumption, therefore the following three properties hold.

Non-singularity of the Time-Optimal Control [1, th.6-5, p.399]: problem 1 is not singular, i.e. the argument of the signum function in Eq. 5 is not identically zero on a finite interval of time. Eq. 5 can be used to find the optimal control for all t in $[t_0, T^*]$. In particular, it yields

$$\mathbf{p}^*(t) = e^{-\mathbf{A}'t}\,\mathbf{p_0}, \tag{6}$$

which substituted into Eq.5 yields

$$\mathbf{u}^*(t) = -sgn\left(\mathbf{B}'e^{-\mathbf{A}'t}\,\mathbf{p_0}\right). \tag{7}$$

The optimal control $\mathbf{u}^*(t)$ is therefore a piecewise constant function taking the values of $+1$ or -1.

Uniqueness of the Time-Optimal Control and of the Extremal Controls: the time-optimal control $\mathbf{u}^*(t)$ is unique [1, th.6-7, p.400]. Furthemore, there exists only one extremal control and is therefore coincident with the optimal control [1, th.6-9, p.404].

Upper Bound on the Number of Switchings: If all of the eigenvalues of $\mathbf{A}$ are real, then the number of switchings of each component of $\mathbf{u}^*(t)$ is at most $n-1$ [1, th.6-8, p.402].

3 Minimum-time control of a linear system between two states having a related offset

This section reports results regarding the minimum-time control of a linear time-invariant normal system between initial and final states which are obtained by offsetting in a specific related way the initial and final states of Problem 1.

Problem 2. ***(Minimum-time control of a linear system between two boundary states which are offset w.r.t. the boundary states of Problem 1)***
Given the linear time-invariant normal system of Eq. 1 with the control constraints of Eq. 2 find the control that transfers the system in a minimum time T_2^* from the initial state,

$$\mathbf{x}(0) = \xi + e^{-\mathbf{A}T_2^*}\phi, \tag{8}$$

where $\phi \in \mathscr{R}^n$ is arbitrary, to the final state

$$\mathbf{x}(T_2^*) = \phi. \tag{9}$$

Notably, in Problem 2 the final state is offset by an arbitrary state ϕ w.r.t. to the origin of the state space. Furthemore, the initial state in Problem 2 is offset by the particular state $e^{-\mathbf{A}T_2^*}\phi$ –related to the offset of the final state[1]– w.r.t. to the original initial state of Problem 1(i.e. ξ).

Here below, basic properties regarding the evolution of a linear system with offset boundary states are stated and proven. These facts are then used to solve Problem 2.

Basic Property 1 *Given the linear time-invariant normal system of Eq. 1, if a control history*

$$\mathbf{U}(t) \in \mathscr{R}^r, \quad t \in [0, t_f] \tag{10}$$

exists such that is bringing the system from an arbitrary initial state

$$\mathbf{x}(0) = \mathbf{x_0}, \tag{11}$$

to an arbitrary final state

$$\mathbf{x}(t_f) = \mathbf{x_f}. \tag{12}$$

in a time $t = t_f$, then the same control history $\mathbf{U}(t)$ is also bringing the system –in the same time $t = t_f$– from the initial state

$$\mathbf{x}(0) = \mathbf{x_0} + e^{-\mathbf{A}t_f}\chi, \tag{13}$$

to the final state

$$\mathbf{x}(t_f) = \mathbf{x_f} + \chi, \tag{14}$$

where $\chi \in \mathscr{R}^n$ is arbitrary.

In other words, in a given time, an identical control history is bringing the system of Eq. 1 from the initial condition in Eq. 11 to the final condition in Eq. 12, and from the initial condition in Eq. 13 to the final condition in Eq. 14.

Proof. This property is an immediate consequence of the principle of effect superposition valid for linear system. In fact, if the effect of the control that brings the system from $\mathbf{x_0}$ to $\mathbf{x_f}$ is added to the effect of the initial condition $e^{-\mathbf{A}t_f}\chi$ which naturally brings the uncontrolled system to χ, then the end result of these two effects superimpose as the system is linear. This conclusion can be put in mathematical terms as follows. The well known solution of a linear-time invariant system governed by Eq. 1 with input $\mathbf{U}(t)$ and initial state $\mathbf{x}_0$ is [6]

$$\mathbf{x}(t) = e^{\mathbf{A}t}\,\mathbf{x_0} + \int_0^t e^{\mathbf{A}(t-s)}\mathbf{B}\,\mathbf{U}(s)\,ds \tag{15}$$

and, in particular, at the final time t_f,

[1] It is immediate to see that the system in Eq. 1 if uncontrolled and subjected to the initial condition $e^{-\mathbf{A}T_2^*}\phi$ will naturally evolve in a period of time T_2^* to the final condition ϕ.

$$\mathbf{x}(t_f) = e^{\mathbf{A}t_f}\,\mathbf{x_0} + \int_0^{t_f} e^{\mathbf{A}(t_f - s)}\mathbf{B}\,\mathbf{U}(s)\,ds = \mathbf{x_f}. \tag{16}$$

If the same control history is applied to the system of Eq. 1 but with initial state as in Eq. 13 it yields

$$\mathbf{x}(t_f) = e^{\mathbf{A}t_f}\left(\mathbf{x_0} + e^{-\mathbf{A}t_f}\chi\right) + \int_0^{t_f} e^{\mathbf{A}(t_f - s)}\mathbf{B}\mathbf{U}(s)\,ds, \tag{17}$$

which, by taking into account Eq. 16, yields

$$\mathbf{x}(t_f) = \mathbf{x_f} + \chi. \tag{18}$$

□

Theorem 1. ***(Solution of Problem 2)***
The optimal control history $\mathbf{u}^*(t)$ expressed by Eq.7, which is the solution of Problem 1 and therefore transfers the system of Eq. 1 from an arbitrary initial state ξ to the state-space origin $\mathbf{0}$ in the minimum time T^*, is also the optimal control solution of the associated Problem 2 of transferring, i.e. is the control which transfers, with minimum time T^*, the same system of Eq. 1 from the initial state

$$\mathbf{x}(0) = \xi + e^{-\mathbf{A}T^*}\phi, \tag{19}$$

where $\phi \in \mathscr{R}^n$ is arbitrary, to the final state

$$\mathbf{x}(T^*) = \phi. \tag{20}$$

Proof. The proof of this theorem is contained in ref. [16].

4 Minimum-time control of a linear system between two arbitrary states

This section reports the problem statement as well as new results regarding the minimum-time control of a linear time-invariant normal system from an arbitrary initial state to an arbitrary desired final state.

Problem 3. ***(Minimum-time control of a linear system between two arbitrary states)***
Given the linear time-invariant normal system of Eq. 1 with the control constraints of Eq. 2 and given an arbitrary state, at the initial time $t = t_0 = 0$,

$$\mathbf{x}(0) = \eta, \tag{21}$$

find the control $\overline{\mathbf{u}}^*(t)$ that transfers the system in a minimum time τ^*, to an arbitrary final state

$$\mathbf{x}(\tau^*) = \phi. \tag{22}$$

Theorem 2. ***(Solution of Problem 3)***
The optimal control history $\overline{\mathbf{u}}^*(t)$ which is the solution of Problem 3 and therefore transfers the system of Eq. 1 from an arbitrary initial state η to an arbitrary final state ϕ in a minimum time τ^*, is the same optimal control history which transfers the system from the particular initial state

$$\mathbf{x}(0) = \xi(\tau^*) = \eta - e^{-\mathbf{A}\tau^*}\phi, \tag{23}$$

to the origin of the state-space

$$\mathbf{x}(\tau^*) = \mathbf{0}, \tag{24}$$

in the same time τ^*,
where τ^*, which determines the particular initial state $\xi(\tau^*)$, is given by

$$\tau^* = \min\{0 \le \tau \in \mathscr{R} \quad : | \quad T^*(\xi(\tau)) = \tau\}, \tag{25}$$

where $T^*(\xi(\tau))$ is the minimum time it takes to transfer the system from the initial condition

$$\mathbf{x}(0) = \xi(\tau) = \eta - e^{-\mathbf{A}\tau}\phi, \tag{26}$$

to the state-space origin.

Proof. The proof of this theorem is contained in ref. [16].

One could intuitively wonder whether Problem 3 could be trivially solved by considering a hypothetically equivalent Problem 1 obtained by relocating the origin of the state space at the desired final state. However this is in the general case not true as demonstrated here below.

Corollary 1. *The problem of transferring in minimum time the system of Eq. 1 from an arbitrary initial state η to an arbitrary final state ϕ (Problem 3) in general* is not equivalent *to the problem obtained by relocating the origin of the state space at the desired final state, i.e. to the problem of transferring in minimum time the following system*

$$\dot{\mathbf{w}}(t) = \mathbf{A}\mathbf{w}(t) + \mathbf{A}\phi + \mathbf{B}\mathbf{u}(t), \tag{27}$$

obtained from the original system of Eq. 1 through the change of state variables

$$\mathbf{x}(t) = \mathbf{w}(t) + \phi, \tag{28}$$

from the initial transformed-state condition

$$\mathbf{w}(0) = \eta - \phi \tag{29}$$

to the final transformed-state condition

$$\mathbf{w}(t_f) = \mathbf{0}. \tag{30}$$

This equivalence is true only for the exceptional case in which the desired final state is an equilibrium state for the original system of Eq. 1. This exceptional case is considered for instance in [4, 111][10, 184–200][17, 218–230].

Proof. If the desired final state ϕ is an equilibrium state for the system in Eq. 1[2]

$$\mathbf{A}\phi = \mathbf{0} \rightarrow \mathbf{e}^{\mathbf{At}}\phi = \phi \quad \forall \mathbf{t}, \tag{31}$$

then the transformed system in Eq. 27 is

$$\dot{\mathbf{w}}(t) = \mathbf{A}\mathbf{w}(t) + \mathbf{B}\mathbf{u}(t), \tag{32}$$

which is equivalent and evolves as the original system in Eq.1, and the initial state in Eq. 23 becomes equivalent to the one in Eq. 29.

Therefore, in this exceptional case, Problem 3 becomes equivalent to a particular instance of Problem 1, with

$$\mathbf{x}(0) = \xi = \eta - \phi. \tag{33}$$

However, in the general case when the desired final state ϕ is *not* an equilibrium state for the system in Eq. 1, i.e.

$$\mathbf{A}\phi \neq \mathbf{0} \rightarrow \mathbf{e}^{\mathbf{At}}\phi \neq \phi \quad \forall \mathbf{t} > \mathbf{0}, \tag{34}$$

then the transformed system in Eq. 27 evolves –differently than the original system in Eq. 1– as follows [6]

$$\mathbf{w}(t) = e^{\mathbf{A}t}\,\mathbf{w_0} + \int_0^t e^{\mathbf{A}(t-s)} ds\, \mathbf{A}\phi + \int_0^t e^{\mathbf{A}(t-s)}\mathbf{B}\,\mathbf{u}(s)\, ds, \tag{35}$$

and the initial state in Eq. 23 is not equivalent to the one in Eq. 29.

The solution of the optimal control problem is in this case the one reported in Theorem 2. □

5 Solved Example: Minimum-time control of the Double Integrator between arbitrary states

A one-d.o.f. *double-integrator* system is governed by the following equation

$$\begin{cases} I\ddot{x}(t) = C(t) \\ |C(t)| \leq C_{\max} \end{cases}, \tag{36}$$

[2] An equilibrium state $\mathbf{x_{eq}}$ is a state at which an unperturbed system stays permanently; for LTI systems this happens when $\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x_{eq}} = \mathbf{x_{eq}}$, i.e. iff $\mathbf{A}\mathbf{x_{eq}} = \mathbf{0}$.

where $C_{\max}$ is the maximum strength of the control and I is the inertia parameter of the dynamics. By a suitable scaling of the control and state variables, the system of Eq. 36 can be equivalently written in state-space form, with obvious meaning of the symbols, as

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \\ |u(t)| \leq 1 \end{cases}, \tag{37}$$

where:

$$\mathbf{x}(t) = \begin{bmatrix} x \\ \dot{x} \end{bmatrix} ; \mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \ \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{38}$$

5.1 *Solution of Problem 1 for the double-integrator*

The system of Eq. 37 is normal, and the time optimal control sequence has at most one switching. Let $x_0 = x(0)$ and $\dot{x}_0 = \dot{x}(0)$ be the initial conditions; the solution is found through a geometrical approach by exploiting the trajectories of the state in the phase plane of equation [3]

$$\left(x - \frac{1}{2}u^*\dot{x}^2\right) = \left(x_0 - \frac{1}{2}u^*\dot{x}_0^2\right) \tag{39}$$

where $u^* = +1$ or $u^* = -1$. Eq. 39 gives the equation of a family of parabolas with the vertex coordinates in $(x_0 - \frac{1}{2}u^*\dot{x}_0, 0)$ and axis of symmetry $\dot{x} = 0$. Moreover, if $u^* = +1$ the concavity is positive, otherwise if $u^* = -1$ the concavity is negative. As result, the *switch curve F* is determined in the phase plane by the equation

$$F(x,\dot{x}) = x + \frac{1}{2}\mathrm{sgn}\{\dot{x}\}\dot{x}^2 = 0 \tag{40}$$

where "sgn" is the signum function.
Assume $F_0 = F(x_0,\dot{x}_0) \neq 0$; the time optimal control sequence transferring the initial condition to the origin of the phase plane is $u_0^* = -\mathrm{sgn}\{F_0\}$ and $u_1^* = -u_0^*$. Starting from $(x_0,\dot{x}_0)$, the state travels along with a branch of parabola forced by u_0^*; when it reaches the *switch curve*, u_1^* transfers the state to the origin along with a branch of parabola on the *switch curve* (see Fig. 1).

The intersection point H of the two branches of parabolas has coordinates in the phase plane

$$\begin{cases} x_s = \dfrac{1}{2}\left(x_0 - \dfrac{1}{2}u_0^*\dot{x}_0^2\right) \\ \\ \dot{x}_s = u_0^*\sqrt{-u_0^*x_0 + \dfrac{1}{2}\dot{x}_0^2} \end{cases} \tag{41}$$

[3] For the sake of simplicity, because $u = +1$ or $u = -1$, in the following in order to avoid the control variable at the denominator, where $1/u$ should appear, u replaces it.

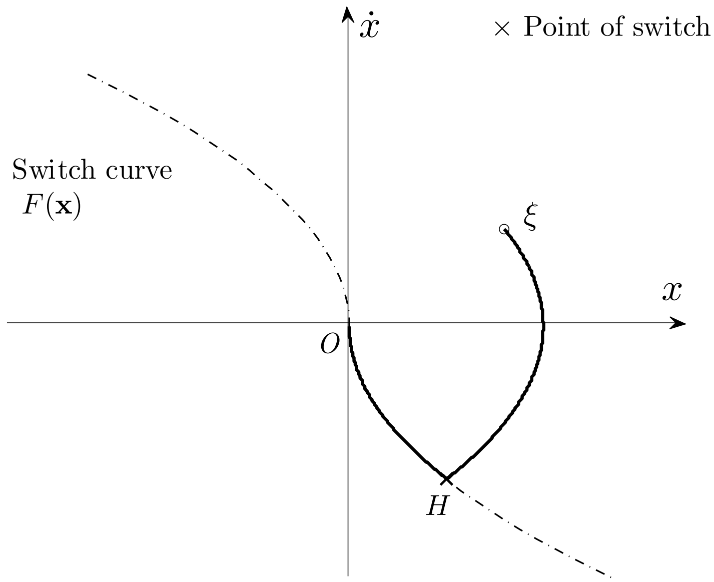

Fig. 1 State trajectory: $\xi(2,1)$, $T^* = 4.16\ s$

Let Δ_0 and Δ_1 be the durations of the control u_0^* and u_1^* respectively; assuming the initial time $t = 0$, and using the solution of the position and velocity as a function of the time, we have:

$$\begin{cases} \Delta_0 = \sqrt{-u_0^* x_0 + \frac{1}{2}\dot{x}_0^2} - u_0^* \dot{x}_0 \\ \\ \Delta_1 = \sqrt{-u_0^* x_0 + \frac{1}{2}\dot{x}_0^2} \end{cases} \tag{42}$$

If the initial condition $(x_0, \dot{x}_0)$ belongs to the *switch curve*, it results in $F_0 = 0$, and no switching occurs. This means that $\Delta_1 = 0$, and because $\Delta_0 = \Delta_1 - u_0^* \dot{x}_0$ needs to be positive (see Eq. 42), the admissible control in minimum time is :

$$u_0^* = -\mathrm{sgn}\{\dot{x}_0\} \tag{43}$$

and the state trajectory is only a part of the switch curve.
Therefore the time-optimal control sequence that transfers any initial state to the origin is:

$$
\begin{aligned}
&(a)\ \ F_0 = 0 \rightarrow u_0^* = -\mathrm{sgn}\{\dot{x}_0\} \\
&\text{with duration}\ \ \Delta_0 = |\dot{x}_0| \\
&(b)\ \ F_0 \neq 0 \rightarrow u_0^* = -\mathrm{sgn}\{F_0\};\ \ u_1^* = -u_0^* \\
&\text{with durations} \\
&\Delta_0 = \sqrt{\frac{1}{2}\dot{x}_0^2 + \mathrm{sgn}\{F_0\}x_0} + \mathrm{sgn}\{F_0\}\dot{x}_0 \\
&\Delta_1 = \sqrt{\frac{1}{2}\dot{x}_0^2 + \mathrm{sgn}\{F_0\}x_0}
\end{aligned}
\tag{44}
$$

For the case (b), $u_0^* = -\mathrm{sgn}\{F_0\}$ has to be kept for a length of time Δ_0, while $u_1^* = -u_0^* = \mathrm{sgn}\{F_0\}$ has to be kept for a length of time Δ_1 in order to reach the origin. Alternatively, the control sequence can be seen as a feedback control. In fact, starting from the initial conditions $(x_0, \dot{x}_0)$, $u_0^* = -\mathrm{sgn}\{F_0\}$ is kept until the state reaches $(x_s, \dot{x}_s)$, while starting from $(x_s, \dot{x}_s)$, $u_1^* = \mathrm{sgn}\{F_0\}$ is kept until the state $(x, \dot{x})$ reaches $(0,0)$.
Finally, the expression of the minimum time T^* is:

$$
\begin{aligned}
&(a)\ \ F_0 = 0: \qquad T^* = |\dot{x}_0| \\
&(b)\ \ F_0 \neq 0: \\
&\qquad T^* = 2\sqrt{\frac{1}{2}\dot{x}_0^2 + \mathrm{sgn}\{F_0\}x_0} + \mathrm{sgn}\{F_0\}\dot{x}_0
\end{aligned}
\tag{45}
$$

Therefore, for the Problem 1 Eq. 44 and Eq. 45 yield the solution in closed-form of the optimal control sequence and the minimum time as a function of the initial condition; that is, when the initial condition belongs to the subset defined by the locus of points in the phase plane for which $F_0 > 0$, $F_0 < 0$ or $F_0 = 0$, then the minimum-time solution is given.

5.2 Solution of Problem 3 for the double-integrator

The solution of Problem 3 for the double-integrator is presented here for the first time to the best knowledge of the authors by exploiting the expressions of Sec. 4 and the solution approach of Sec. 5.1. Let us call η the state with components in the phase plane $(x_0, \dot{x}_0)$, to be transferred in minimum time to the state ϕ with components in the phase plane $(x_f, \dot{x}_f)$.

Theorem 3. ***(Minimum-time control of the double-integrator between arbitrary states):*** *Let us define the function:*

$$
\overline{F}_0 = (x_0 - x_f) + \mathrm{sgn}\{\dot{x}_0 - \dot{x}_f\}\frac{(\dot{x}_0^2 - \dot{x}_f^2)}{2}. \tag{46}
$$

The optimal control sequence transferring in the minimum-time τ^* the system of Eq. 37 from an arbitrary initial state η to an arbitrary final state ϕ is:
(a) if $\overline{F}_0 = 0$

$$\overline{u}_0^* = -sgn\{\dot{x}_0 - \dot{x}_f\} \tag{47}$$

with time duration $\Delta_0 = |\dot{x}_0 - \dot{x}_f|$ *and* $\tau^* = \Delta_0$*;*

(b) if $\overline{F}_0 \neq 0$

$$\overline{u}_0^* = -sgn\{\overline{F}_0\}; \;\; \overline{u}_1^* = -\overline{u}_0^* \tag{48}$$

with time durations respectively

$$\begin{aligned} \Delta_0 &= \sqrt{\frac{\dot{x}_0^2 + \dot{x}_f^2}{2} + \mathrm{sgn}\{\overline{F}_0\}(x_0 - x_f)} + \mathrm{sgn}\{\overline{F}_0\}\dot{x}_0 \\ \Delta_1 &= \sqrt{\frac{\dot{x}_0^2 + \dot{x}_f^2}{2} + \mathrm{sgn}\{\overline{F}_0\}(x_0 - x_f)} + \mathrm{sgn}\{\overline{F}_0\}\dot{x}_f \end{aligned} \tag{49}$$

and $\tau^* = \Delta_0 + \Delta_1$.

Proof. Theorem 2 states that the optimal control sequence transferring in the minimum-time τ^* the system of Eq. 37 from an arbitrary initial state η to an arbitrary final state ϕ is the same control sequence that transfers with the same minimum-time τ^* the same system of Eq. 37 from the particular initial state $\xi(\tau)$ to the origin of the phase plane, with $\tau = \tau^*$ solution of Eq. 25, where for the double integrator

$$e^{-\mathbf{A}\tau} = \begin{pmatrix} 1 & -\tau \\ 0 & 1 \end{pmatrix} \tag{50}$$

Let $(\xi, \dot{\xi})$ be the components in the phase plane of ξ and $(\tilde{\xi}, \dot{\tilde{\xi}})$ be the components of $\tilde{\xi} = \xi(0) = \eta - \phi$; as a result we have

$$\begin{aligned} \xi &= (x_0 - x_f) + \dot{x}_f\tau = \tilde{\xi} + \dot{x}_f\tau \\ \dot{\xi} &= (\dot{x}_0 - \dot{x}_f) = \dot{\tilde{\xi}} \end{aligned} \tag{51}$$

Therefore $\xi(\tau)$ belongs to a line parallel to the abscissa of the phase plane. The particular initial condition $\xi(\tau^*)$ is found by increasing the value of τ until it reaches the value τ^* that satisfies the equation $T^*(\xi(\tau)) = \tau$.

Let us first assume that $\xi(\tau^*)$ is a point of the switch curve; according to the solution of the Problem 1 the minimum time control does not have any switching and using Eq. 44 and Eq. 45 we have

$$\begin{aligned} u_0 &= -\mathrm{sgn}\{\dot{\tilde{\xi}}\} = -\mathrm{sgn}\{\dot{x}_0 - \dot{x}_f\} \\ \tau^* &= |\dot{\tilde{\xi}}| = |\dot{x}_0 - \dot{x}_f|. \end{aligned} \tag{52}$$

Thus the particular initial condition in this case is the point with components

$$\begin{aligned} \xi &= \tilde{\xi} + \dot{x}_f|\dot{x}_0 - \dot{x}_f| = \tilde{\xi} + \mathrm{sgn}\{\dot{x}_0 - \dot{x}_f\}(\dot{x}_0\dot{x}_f - \dot{x}_f^2) \\ \dot{\tilde{\xi}} &= (\dot{x}_0 - \dot{x}_f). \end{aligned} \tag{53}$$

Using the expressions of Eq. 53 in the equation of the switch curve Eq. 40

$$F(\xi,\dot{\xi}) = \tilde{\xi} + \mathrm{sgn}\{\dot{\tilde{\xi}}\}\left[(\dot{x}_0\dot{x}_f - \dot{x}_f^2) + \frac{1}{2}\dot{\tilde{\xi}}^{\,2}\right] = 0 \tag{54}$$

that it, in turn, results in

$$(x_0 - x_f) + \mathrm{sgn}\{\dot{x}_0 - \dot{x}_f\}\frac{(\dot{x}_0^2 - \dot{x}_f^2)}{2} = \overline{F}_0 = 0. \tag{55}$$

This part concludes the demonstration of the case (a) of the theorem.
If $\xi(\tau^*)$ does not belong to the switch curve, it means that $\overline{F}_0 \neq 0$; as a consequence, according to Eq. 44, the time optimal sequence is given by Eq. 48 with time durations

$$\begin{aligned} \Delta_0 &= \Delta_1 + \mathrm{sgn}\{\overline{F}_0\}\dot{\tilde{\xi}} \\ \Delta_1 &= \sqrt{\frac{1}{2}\dot{\tilde{\xi}}^2 + \mathrm{sgn}\{\overline{F}_0\}\left(\tilde{\xi} + \dot{x}_f\tau^*\right)}. \end{aligned} \tag{56}$$

It is worth noting that Δ_0 and Δ_1, by definition, are both positive and $\tau^* = \Delta_0 + \Delta_1$ as stated in Theorem 2. For the sake of clarity, let us name $s_0 = \mathrm{sgn}\{\overline{F}_0\}$. As result, Δ_1 can be found as a solution of the second order algebraic equation

$$\Delta_1^2 - 2s_0\dot{x}_f\Delta_1 - \frac{(\dot{x}_0^2 - \dot{x}_f^2)}{2} - s_0(x_0 - x_f) \tag{57}$$

that has roots

$$\Delta_1 = \pm\sqrt{\frac{(\dot{x}_0^2 + \dot{x}_f^2)}{2} + s_0(x_0 - x_f)} + s_0\dot{x}_f \tag{58}$$

and yielding (see Eq. 44)

$$\Delta_0 = \pm\sqrt{\frac{(\dot{x}_0^2 + \dot{x}_f^2)}{2} + s_0(x_0 - x_f)} + s_0\dot{x}_0. \tag{59}$$

Therefore, the proper value of τ^* is found by the proper selection of the sign in Eq. 58 and Eq. 59. Notice that, because Δ_0 and Δ_1 must be both positive, when $\dot{x}_0 = 0$ and, or, $\dot{x}_f = 0$, the only admissible choice of the sign is "+". For this reason, in the following we assume $\dot{x}_0 \neq 0$ and $\dot{x}_f \neq 0$. Exploiting the expression of $\overline{F}_0$ we need to distinguish two cases: $s_0 = +1$ and $s_0 = -1$.
If $s_0 = +1$, we have for $\dot{\tilde{\eta}} = (\dot{x}_0 - \dot{x}_f) \geq 0$

$$\begin{aligned} \Delta_0 &= \pm\delta|\dot{x}_f| + \dot{x}_0 = \pm\delta|\dot{x}_f| + \dot{x}_f + \dot{\tilde{\eta}} \\ \Delta_1 &= \pm\delta|\dot{x}_f| + \dot{x}_f \end{aligned} \tag{60}$$

with $\delta = \sqrt{1 + \overline{F}_0/\dot{x}_f^2} > 1$, while for $\dot{\tilde{\eta}} = (\dot{x}_0 - \dot{x}_f) < 0$

$$\begin{aligned} \Delta_0 &= \pm\delta'|\dot{x}_0| + \dot{x}_0 \\ \Delta_1 &= \pm\delta'|\dot{x}_0| + \dot{x}_f = \pm\delta'|\dot{x}_0| + \dot{x}_0 - \dot{\tilde{\eta}} \end{aligned} \tag{61}$$

where $\delta' = \sqrt{1+\overline{F}_0/\dot{x}_0^2} > 1$.
If $s_0 = -1$, we have for $\dot{\tilde{\eta}} = (\dot{x}_0 - \dot{x}_f) \geq 0$

$$\begin{array}{l} \Delta_0 = \pm\sigma|\dot{x}_0| - \dot{x}_0 \\ \Delta_1 = \pm\sigma|\dot{x}_0| - \dot{x}_f = \pm\sigma|\dot{x}_0| - \dot{x}_0 + \dot{\tilde{\eta}} \end{array} \tag{62}$$

with $\sigma = \sqrt{1+|\overline{F}_0|/\dot{x}_0^2} > 1$, while for $\dot{\tilde{\eta}} = (\dot{x}_0 - \dot{x}_f) < 0$

$$\begin{array}{l} \Delta_0 = \pm\sigma'|\dot{x}_f| - \dot{x}_0 = \pm\sigma'|\dot{x}_f| + \dot{x}_f - \dot{\tilde{\eta}} \\ \Delta_1 = \pm\sigma'|\dot{x}_f| - \dot{x}_f \end{array} \tag{63}$$

where $\sigma' = \sqrt{1+|\overline{F}_0|/\dot{x}_f^2} > 1$.
In order to have always $\Delta_0 > 0$ and $\Delta_1 > 0$, the above relations state that for any value of $\operatorname{sgn}\{\dot{x}_0\}$ and $\operatorname{sgn}\{\dot{x}_f\}$ the only admissible sign in Eq. 58 and Eq. 59 is "+".
□

Corollary 2. *The switch curve of the minimum-time control for the double-integrator transferring any initial state* $\mathbf{x}(x,\dot{x})$ *to an arbitrary final state* $\phi(x_f\dot{x}_f)$*, has the expression*

$$\overline{F}(\mathbf{x},\phi) = (x - x_f) + \operatorname{sgn}\{\dot{x} - \dot{x}_f\}\frac{(\dot{x}^2 - \dot{x}_f^2)}{2} = 0 \tag{64}$$

Proof. In the demonstration of Theorem 3 , Eq. 55 shows the relation of the initial and final conditions on a point of the switch curve. Assuming any initial condition in the phase plane, it results in Eq. 64. □

It is worth noting that Eq. 64 becomes the known expression of the switch curve Eq. 40, when the final state is the origin of the phase plane.
Figure 2 shows the state trajectory from an arbitrary initial state η to an arbitrary final state ϕ, while Figure 3 shows the time optimal state trajectory when the two arbitatry states are switched. Moreover, the figures depict the time optimal state trajectory of the equivalent Problem 1 starting from the particular initial condition ξ (bold dashed lines) and the searching path that $\xi(\tau)$ travels when τ moves from $\tau = 0$ to $\tau = \tau^*$ (arrowed dotted line). Starting from the initial conditions ξ, $\overline{u}_0^*$ is kept until the state reaches the point P$(\overline{x}_s, \dot{\overline{x}}_s)$, with:

$$\begin{cases} \overline{x}_s = \dfrac{1}{2}\left(x_0 + x_f - \overline{u}_0^* \dfrac{\dot{x}_0^2 - \dot{x}_f^2}{2}\right) \\ \\ \dot{\overline{x}}_s = \overline{u}_0^* \sqrt{-\overline{u}_0^*\left(x_0 - x_f\right) + \dfrac{\dot{x}_0^2 + \dot{x}_f^2}{2}} \end{cases} \tag{65}$$

while from $(\overline{x}_s, \dot{\overline{x}}_s)$, $\overline{u}_1^*$ is kept until the state reaches ϕ. Therefore, for the Problem 2, Theorem 3 yields the solution in closed-form of the optimal control sequence and the minimum time as a function of the initial and final conditions; that is, when

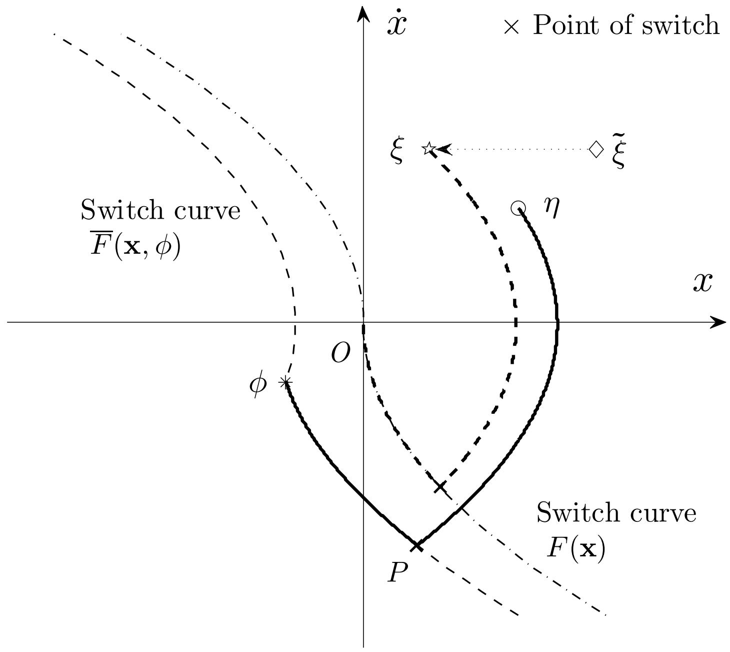

Fig. 2 State trajectory: $\eta(2,1)$, $\phi(-1,-0.5)$, $\tau^* = 4.31\ s$

the initial condition belongs to the subset defined by the locus of points in the phase plane for which $\overline{F}_0 > 0$, $\overline{F}_0 < 0$ or $\overline{F}_0 = 0$, then the minimum-time solution is found.

6 Conclusions

In this paper, for the first time to the best knowledge of the authors, a solution method is presented for the problem of minimum-time control of a general linear time-invariant normal system evolving from an arbitrary initial state to an arbitrary desired final state subjected to a cubic-constrained control. The method is then applied to solve, as important example, the case of the double-integrator.

The newly found solutions here presented have a high theoretical importance since they add elements to the small set of existing closed-form solutions of optimal control problems, as well as a practical importance since they can be used as benchmark comparison cases for numerical optimal control solvers and to build approximate solutions for more complex cases.

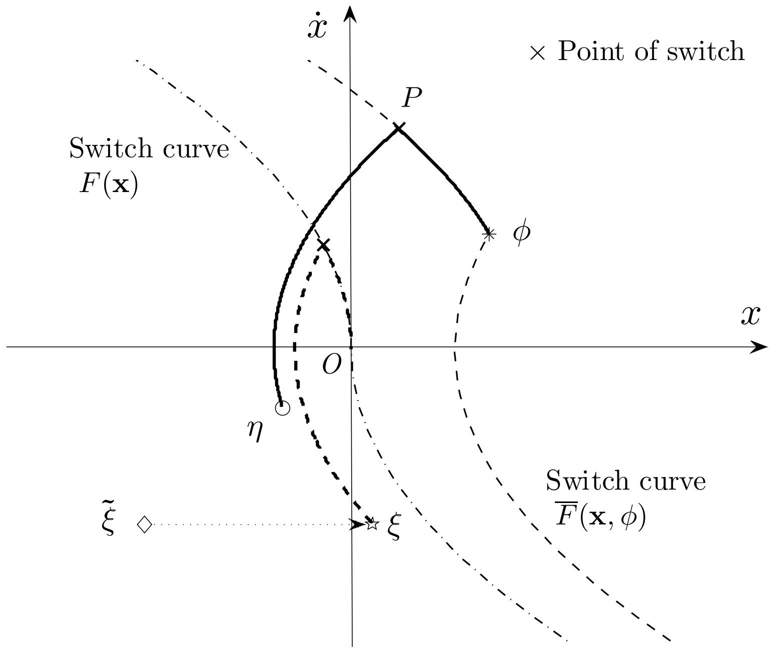

Fig. 3 State trajectory: $\eta(-1,-0.5)$, $\phi(2,1)$, $\tau^* = 3.31\ s$

References

1. M. Athans and P. L. Falb. *Optimal Control. An Introduction to the Theory and Its Applications*. McGraw-Hill, 1966.
2. R. Bellman, editor. *Mathematical Optimization Techniques*. The RAND Corporation, 1963.
3. V.G. Boltyanskii. *Mathematical Methods of Optimal Control*. Holt, Rinehart and Winston, Inc., 1971.
4. A.E. Bryson and Y.C. Ho. *Applied Optimal Control*. Blaisdell Publishing Company, 1969.
5. D.W. Bushaw. *Differential equations with a discontinuous forcing term*. Doctoral Dissertation, Princeton, 1952.
6. C.T. Chen. *Linear System Theory and Design*. Oxford University Press, 1999.
7. S. Choura and S. Jayasuriya. An explicit minimum-time solution for the simple harmonic oscillator. *Journal of Dynamics Systems, Measurement and Control*, 112:783–787, 1990.
8. Andrea Garulli, Antonio Giannitrapani, and Mirko Leomanni. Minimum switching control for systems of coupled double integrators. *Automatica*, 60:115 – 121, 2015.
9. H.P. Geering. *Optimal Control with Engineering Applications*. Springer, Berlin, Heidelberg, 2007.
10. J.L. Junkins and J.D. Turner. *Optimal Spacecraft Rotational Maneuvers*. Elsevier, 1986.
11. D.E. Kirk. *Optimal Control Theory: An Introduction*. Dover, 1998.
12. E.B. Lee and L. Markus. *Foundations of Optimal Control Theory*. Wiley & Sons, Inc., 1967.
13. G. Leitmann. *The calculus of variations and optimal control: an introduction*. Plenum Press, 1981.
14. L.S. Pontryagin, V.G. Boltyanskii, R.V. Gamkrelidze, and E. Mishchenko. *The mathematical theory of optimal processes (International series of monographs in pure and applied mathematics)*. Interscience Publishers, 1962.

15. V. G. Rao and D. S. Bernstein. Naive control of the double integrator. *IEEE Control Systems*, 21(5):86–97, Oct 2001.
16. M. Romano and F. Curti. *Time-Optimal Control of Linear Time-invariant Systems between two Arbitrary States*. Submitted for possible publication.
17. I.M. Ross. *A Primer on Pontryagin's Principle in Optimal Control*. Collegiate Publishers, 2015.
18. A.P. Sage and C.C. White. *Optimum Systems Control*. Prentice-Hall Inc., 1977.
19. S. Scaramozzino, K.D. Listmann, and J. Gebhardt. Time-optimal control of harmonic oscillators at resonance. In *European Control Conference (ECC)*, 2015.
20. Edoardo Serpelloni, Manfredi Maggiore, and Christopher Damaren. Bang–bang hybrid stabilization of perturbed double-integrators. *Automatica*, 69:315 – 323, 2016.
21. S. Emre Tuna. Synchronization of harmonic oscillators under restorative coupling with applications in electrical networks. *Automatica*, 75:236 – 243, 2017.