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Backstepping control for state constrained systems

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Abstract A state constrained control design problem is addressed in this article via Lyapunov techniques. We show that for systems in a special linear strict feedback form, it is possible to impose constraints on states unmatched with the control using a backstepping technique while achieving the stabilization objective. Initially in this article, we formalize for linear systems an existing procedure which uses backstepping control to constrain partial states of a spacecraft's attitude dynamics [SB15]. We further show that an extension of the method allows us to constrain all the states of the system simultaneously. In contrast to existing methods using Barrier Lyapunov functions, our controller does not result in large control actions close to the boundary of the convex constraint. Sample simulations are shown to illustrate our theoretical results.

1 Introduction

Most practical controlled dynamical systems require implementation of control and state magnitude constraints. For example, in mechanical and chemical applications involving pumps or valves, the capacities and range of positions are finite and subject to design. Control algorithms based on classical constraint free design that exceed these limits are therefore rendered useless and new techniques need to be evolved.

Constrained stabilization can be implemented using control design for stabilizable linear systems, whose domain of attraction can be approximated by a polyhedron [BM96]. Related research on stabilization for semi-globally or globally con-

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strained systems also exists [SSSS]. Since these formulations can be rarely implemented online, we aim to generate a simpler law which can achieve results with considerably less computational effort.

More recently it has been shown that it is possible to implement state constrained control using 'Barrier' Lyapunov functions [TG09, NMJ05] for systems in strict feedback form by using backstepping based design. Also, control barrier functions (CBF) and control Lyapunov functions (CLF) can be combined to enforce state-dependent constraints [NS16, AXGT17]. Asymptotic tracking can be achieved without violation of the constraints with a soft condition on the initial output [TGT09]. However, the corresponding feasibility conditions, which are obtained by solving a static optimization problem, can be highly restrictive with respect to the initial states and control parameters. Sample applications include spacecraft reorientation in the presence of attitude constraints by utilizing convex parameterizations [LM11, LM12]. However, the designed control input is typically very high close to the constraint boundary due to the nature of Barrier Lyapunov functions which map a compact constraint set to $[0,\infty)$.

In this article we propose standard backstepping techniques as a means of constraining a partial set of states and further generalize the method to constrain all the states of a linear system in a special feedback form. We therefore avoid Barrier Lyapunov constructions and demonstrate lower control efforts in comparative simulations. The article is organized in four sections. In section 2, state constrained control is designed for constraints on unmatched states. This is then extended to full state constraints in section 3. Simulation studies are presented in section 4 for comparison with existing results.

Notation:

- A^{-R} is the right-inverse of matrix A.
- Similarly A^{-L} denotes the left-inverse of A.
- $|\mu|$ is the absolute value of the scalar μ.
- $||\cdot||$ refers to the 2-norm on vectors and the corresponding induced norm on matrices.
- $\lambda_{min}(A)$ and $\lambda_{max}(A)$ denote the smallest and largest eigenvalues of a real square matrix A respectively.
- $I_n \in \mathbb{R}^{n\times n}$ denotes the identity matrix of dimension $n \times n$.

1.1 Preliminaries:

We employ backstepping tools which can be implemented for control design for autonomous, nonlinear systems in the following *strict-feedback form* [KKK95, sec. 2.3.1], [Kha02, pg. 595]:

$$\begin{aligned}
\dot{x} &= f(x) + g(x)\xi_1 \\
\dot{\xi}_1 &= f_1(x,\xi_1) + g_1(x,\xi_1)\xi_2 \\
&\vdots \\
\dot{\xi}_k &= f_k(x,\xi_1,\ldots,\xi_k) + g_k(x,\xi_1,\ldots,\xi_k)u
\end{aligned}$$

where $x \in \mathbb{R}^n$ and $\xi_1,\ldots,\xi_k$ are scalars and $f(0) = 0$. It is typically assumed that there exists a continuous feedback control law corresponding to ξ_1 as the control input,

$$\xi_{1,desired} = \alpha(x), \alpha(0) = 0$$

and a smooth, positive definite, radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}$ such that

$$\frac{\partial V}{\partial x}(x)[f(x) + g(x)\alpha(x)] \le -W(x) \le 0, \forall x \in \mathbb{R}^n$$

where $W : \mathbb{R}^n \to R$ is positive definite.

Inspired by the above formulation, we now introduce a *linear block strict-feedback form* for autonomous linear systems as below,

$$\begin{aligned}
\dot{x} &= Ax + By, \quad x(0) = x_0, \\
\dot{y} &= Cx + Dy + Eu, \quad y(0) = y_0
\end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ are the states of the system with $u \in \mathbb{R}^p$ being the control. The quintuple (A,B,C,D,E) are real matrices of appropriate dimensions. (1) can also be written in a block strict-feedback form as follows,

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & E \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} \tag{2}$$

2 Constraints on unmatched states

The control objective of the current section is to constrain the control unmatched states x in dynamics (1) while achieving stabilization of all states to the origin.

We generalize a previously implemented procedure on spacecraft attitude dynamics [SB15] to systems in linear strict feedback form (1) to achieve the aforementioned control objectives.

Theorem 1. *Consider the system dynamics* (1) *along with the assumption that* (A,B) *is a stabilizable pair and* E *has a right inverse, denoted* E^{-R}*. The system is uniformly exponentially stabilized at the origin with the control law given by,*

$$u = E^{-R}[-2\sigma B^T P^T x - ((C - DK + K\tilde{A})x + (D + KB)\tilde{y}) - L\tilde{y}] \tag{3}$$

where,

i. $\tilde{y} := y + Kx$ *is a transformation of the control matched states with* $K \in \mathbb{R}^{m\times n}$ *being a stabilizing gain matrix,*
ii. $L \in \mathbb{R}^{m\times m}$ *is a symmetric positive definite matrix and* $\sigma > 0$*,*
iii. $P \in \mathbb{R}^{n\times n}$ *is the symmetric positive definite solution of the Lyapunov equation* $\tilde{A}^T P + P\tilde{A} = -Q$ *with* $\tilde{A} = A - BK$ *and* Q *being a symmetric positive definite matrix.*

Furthermore, the control law (3) *guarantees that the states x remain constrained as,* $\|x\| \le \alpha$ *for some* $\alpha > 0$ *satisfying* $\lambda_{min}(P)\alpha^2 > x(0)^T Px(0)$ *if the gain* σ *is chosen as,*

$$\sigma \ge \frac{\tilde{y}(0)^T \tilde{y}(0)}{2(\lambda_{min}(P)\alpha^2 - x(0)^T Px(0))} \tag{4}$$

Proof. We utilize a backstepping strategy to stabilize the dynamics (1) at the origin. Assuming y as the virtual control for the x-subsystem in (1), we can design a desired stabilizing virtual feedback $y_d = -Kx$, where the gain matrix K is chosen so as to render $\tilde{A} := (A - BK)$ Hurwitz. We now define, $\tilde{y} := y - y_d$. The transformed dynamics can now be written as,

$$\begin{aligned} \dot{x} &= \tilde{A}x + B\tilde{y} \\ \dot{\tilde{y}} &= (C - DK + K\tilde{A})x + (D + KB)\tilde{y} + Eu. \end{aligned} \tag{5}$$

In order to complete the stability proof, consider a positive definite candidate Lyapunov function of the form,

$$V = x^T Px + \frac{1}{2\sigma}\tilde{y}^T \tilde{y} > 0$$

The directional derivative of V along the dynamics (5) is,

$$\dot{V} = x^T(\tilde{A}^T P + P\tilde{A})x + \tilde{y}^T(B^T P + B^T P^T)x + \frac{\tilde{y}^T}{\sigma}\dot{\tilde{y}}$$

Upon substituting the controller satisfying (3) in dynamics (5), the cross terms in $\dot{V}$ are canceled and we obtain a negative definite $\dot{V}$:

$$\dot{V} = -x^T Qx - \frac{1}{\sigma}\tilde{y}^T L\tilde{y} < 0$$

Therefore, by the Lyapunov theorem [Vid02, pg. 171] we can claim uniform exponential stability of the origin.

The choice of σ as in (4) ensures that,

$$x(0)^T Px(0) + \frac{1}{2\sigma}\tilde{y}(0)^T \tilde{y}(0) \le \lambda_{min}(P)\alpha^2$$

Further, employing the fact that $V(t) < V(0)$ ($\dot{V} < 0$) we obtain,

$$x(t)^T Px(t) + \frac{1}{2\sigma}\tilde{y}(t)^T\tilde{y}(t) \leq \lambda_{min}(P)\alpha^2$$

and hence, $x(t)^T Px(t) \leq \lambda_{min}(P)\alpha^2$. The following inequality on quadratic forms,

$$\lambda_{min}(P)||x(t)||^2 \leq x(t)^T Px(t) \leq \lambda_{max}(P)||x(t)||^2 \tag{6}$$

yields,

$$\begin{aligned} \lambda_{min}(P)||x(t)||^2 &\leq \lambda_{min}(P)\alpha^2 \\ \implies ||x(t)|| &\leq \alpha \end{aligned}$$

which guarantees the required bounds on states x.

□

Remark 1. In typical applications, the state constraint α is prescribed by physical limitations and therefore the need to satisfy a condition of the form $\lambda_{min}(P)\alpha^2 > x(0)^T Px(0)$ seems rather stringent. However, given actual constraint on the state, $\bar{\alpha}$, suppose we choose an $\varepsilon > 0$ such that $\bar{\alpha} = (\bar{\alpha} - \varepsilon)\sqrt{\lambda_{max}(P)/\lambda_{min}(P)}$. Now if, $\|x(0)\| < (\bar{\alpha} - \varepsilon)$, then $x(0)^T Px(0) < \lambda_{min}(P)\bar{\alpha}^2$. This requirement, in essence, restricts the initial conditions to a *proper* subset of the constraint set on the states x. It is evident that $\varepsilon \to 0$ as $P \to cI_n$ for $c \in \mathbb{R}$.

Remark 2. The scalar gain σ directly influences the control magnitude in (3). It is therefore paramount to minimize the lower bound on σ in (4). This is achieved by solving the following constrained optimization problem to prescribe the feedback gain K.

$$\begin{aligned} &\min_{K \in \mathbb{R}^{m \times n}} [y(0) + Kx(0)]^T [y(0) + Kx(0)] \\ &\text{such that, } \mathrm{Re}(\lambda(A - BK)) < 0. \end{aligned}$$

This ability to reduce the gain σ distinguishes the current approach from Barrier Lyapunov methods which lead to large control effort close to the constraint boundary.

3 Constraints on all states

The problem of constraining all the states of a system can be reformulated as an unmatched state constraint problem by augmenting the system with additional states. Subsequently, theorem 1 can be directly applied to the augmented system.

Theorem 2. *Consider the linear time-invariant system below:*

$$\dot{x} = Ax + Bu, \quad x(0) = x_0 \tag{7}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and (A,B) stabilizable. Then a dynamic control law defined as below:

$$\dot{u} = Cx + Du + Ev, \quad u(0) = 0 \tag{8}$$

where E is an arbitrary right invertible matrix and $v \in \mathbb{R}^p$ is defined as,

$$v = E^{-R}[-2\sigma B^T P^T x - ((C - DK + K\tilde{A})x + (D + KB)\tilde{u}) - L\tilde{u}] \tag{9}$$

with,

i. $\tilde{u} = u + Kx$,
ii. $L \in \mathbb{R}^{m\times m}$ *is a symmetric positive definite matrix and* $\sigma > 0$,
iii. $P \in \mathbb{R}^{n\times n}$ *is the symmetric positive definite solution of the Lyapunov equation* $\tilde{A}^T P + P\tilde{A} = -Q$ *with* $\tilde{A} = A - BK$ *and Q being a symmetric positive definite matrix.*

guarantees global exponential stability of $(x,u) = (0,0)$. Furthermore, the control law (8)-(9) guarantees that the states x remain constrained as, $\|x\| \leq \alpha$ for some $\alpha > 0$ satisfying $\lambda_{min}(P)\alpha^2 > x(0)^T Px(0)$ if the gain σ is chosen as,

$$\sigma \geq \frac{\tilde{u}(0)^T \tilde{u}(0)}{2(\lambda_{min}(P)\alpha^2 - x(0)^T Px(0))}$$

Proof. The augmented dynamics (7)-(8) have precisely the same linear strict block feedback form considered in theorem 1. Therefore a direct application of theorem 1 proves the claims above. □

4 Simulations

In this section we look at some numerical examples to illustrate the feedback strategies previously presented.

4.1 Constraints on unmatched states

As an example consider the third order integrator dynamics with $x := [x_1, x_2]^T$ and $y := x_3$ in (1), and the following system matrices:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = [0,0], D = [0], E = [1]$$

The initial conditions and constraints are taken to be,

$$\alpha = 12$$
$$x(0) = [11,2]^T,\ y(0) = -4,$$

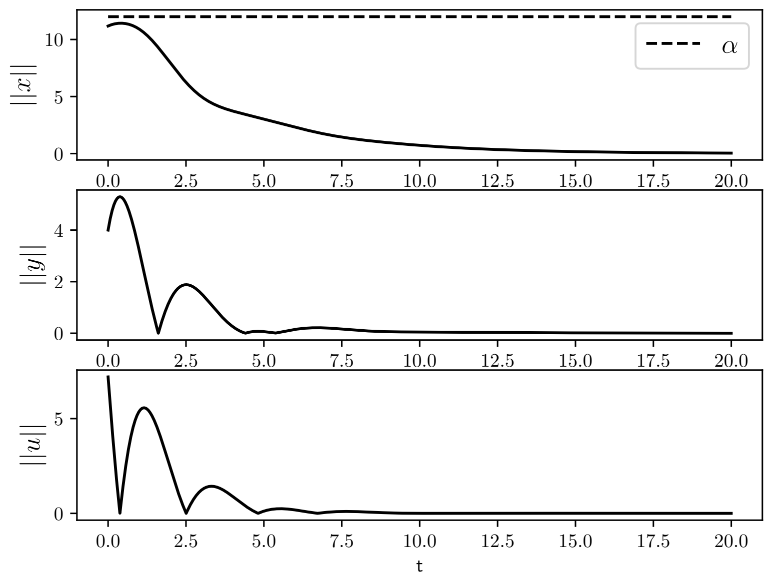

Fig. 1 Constrained dynamics of unmatched states

In order to make all the eigenvalues of P nearly equal and reduce the lower bound on σ, we minimize the weighted average of the ratio of the maximum eigenvalue to the minimum eigenvalue : $\dfrac{\lambda_{max}(P)}{\lambda_{min}(P)}$ and $[y(0)+Kx(0)]^T[y(0)+Kx(0)]$. A Genetic Algorithm (GA) is used which randomly initializes the elements of Q and K and then uses evolutionary operators like selection, crossover and mutation to improve performance of the objective function. In this evolution process, Q has to be positive definite and $\tilde{A} = A - BK$ is required to be Hurwitz. The GA generated values are:

$$Q = -\begin{bmatrix} -50.55 & 217.47 \\ 217.47 & -935.46 \end{bmatrix}$$
$$K = [0.638, 0.639]$$

Further, the gain $L = 1$, and the selected value of σ computed using (4) is 1.027e−3.

As expected by theorem 1 we observe in fig. 1 that the system is exponentially stabilized to the origin and $||x||$ remains bounded by α as expected while maintaining reasonable control magnitudes.

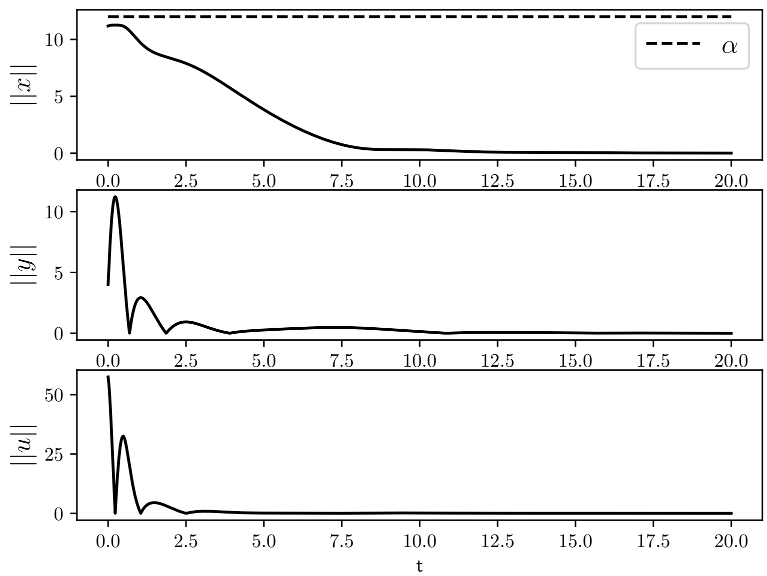

Fig. 2 Constrained dynamics using barrier Lyapunov function

Comparison

We compare our results with the corresponding Barrier Lyapunov based control design. For the system (5), we consider a positive definite candidate Lyapunov (Barrier) function of the form,

$$V = x^T P x \log\left(\frac{tol}{tol - x^T P x}\right) + \frac{1}{2\sigma}\tilde{y}^T \tilde{y} > 0$$

where $tol = \lambda_{min}(P)\alpha^2$. It is evident that the above function maps a compact domain in one argument $\{x \in \mathbb{R}^n | x^T P x \leq tol\} \times \mathbb{R}^m$ to $[0, \infty)$ as required for Barrier functions [NS16]. Upon substituting the following controller,

$$u = E^{-R}[-((C - DK + K\tilde{A})x + (D + KB)\tilde{y}) - L\tilde{y} \\ - 2\sigma\left(\frac{x^T P x}{tol - x^T P x} + \log\left(\frac{tol}{tol - x^T P x}\right)\right) B^T P^T x]$$

in dynamics (5), we obtain a negative definite directional derivative of V,

$$\dot{V} = -x^T Q x \left(\frac{x^T P x}{tol - x^T P x} + \log\left(\frac{tol}{tol - x^T P x}\right)\right) - \frac{1}{\sigma}\tilde{y}^T L \tilde{y}$$

Therefore, by the Lyapunov theorem we can claim uniform exponential stability of the origin. Moreover, since the argument of the logarithm has to be positive, the unmatched states will be bounded as required.

Simulations are carried out for an identical third integrator system with the same initial conditions and parameters. The results can be seen in fig. 2. We observe that thought the states behave similar to our controller, the control effort is an order of magnitude larger with the Barrier function based controller as conjectured. The backstepping controller proposed here allows minimization of the control gain σ by reducing the contribution of unconstrained initial conditions as much as possible by solving a one-time constrained optimization problem to choose gain K. These comparative results are therefore not an isolated phenomenon and have been tested in a large number of examples.

4.2 Constraints on all states

For this case consider a second order integrator system with $x = [x_1, x_2]^T$ with u as the dynamic controller designed as in (8)-(9) with the following system matrices:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = [0, 0], D = [0], E = [1]$$

The initial conditions and constraints are taken to be,

$$\alpha = 12$$
$$x(0) = [11, 2]^T,\ u(0) = 0$$

Using similar optimization techniques as in section 4.1 we design,

$$Q = -\begin{bmatrix} -8.36 & 33.11 \\ 33.11 & -131.07 \end{bmatrix}$$
$$K = [0.440, 0.776]$$

Also, $L = 1$ and the selected value of σ is 0.1015.

As expected from theorem 2 we observe in fig. 3 that the system is exponentially stablilized at the origin and $||x||$ remains bounded by α as expected.

5 Conclusions

We design feedback controllers to constrain some or all states of a dynamical system in a special linear strict feedback form. A backstepping approach is employed to design the feedback law and a constrained optimization problem is solved at ini-

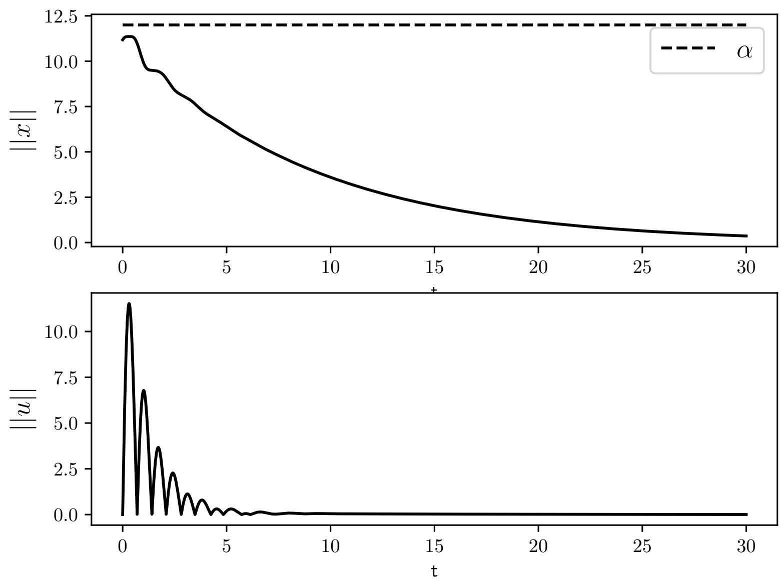

Fig. 3 Full state constrained dynamics

tial time to choose the control gains. This ensures that the states satisfy bounds imposed by the application while keeping the control at reasonable values unlike Barrier Lyapunov constructions in literature. This is also illustrated through exemplary simulations.

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