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Eigenstructure Assignment Accounting for Eigenmode Participation of Control Inputs

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ABSTRACT

Eigenstructure Assignment is a powerful design technique for full state or output feedback based MIMO control systems. It is possible to shape not only the system's closed loop dynamics, but also its eigenvectors - at least partially to an extent depending on the number of available control inputs. Common treatments of the method in literature further reveal a solution to the problem, when the desired eigenstructure refers to the systems output directions (i.e. the amount to which certain output values participate on the various eigenmodes) rather than to the eigenvectors themself directly. However, due to the cross coupling nature of the gain matrix, the physical system inputs are still influenced by multiple eigenmodes, even if the outputs are perfectly decoupled. This may be unfavorable, if certain dynamic properties of the actuation inputs are required (e.g. a low frequent change of thrust input). In this work the concept of eigenstructure assignment is extended to the input variables. The described method allows for a flexible formulation of structural requirements in terms of input and output variables in arbitrary combination.

Keywords: Eigenstructure Assignment; Input Decoupling; Flight Control

1 Introduction

Although the roots of eigenstructure assignment technique reach back to the exploration of vibrating strings modal shapes and is linked to the concept of generalized coordinates, the theory behind has been exhaustively treated and developed to maturity within the late 1970's and early 1980's. This inolves studying the available degree of freedom beyond pole placement [1, 2], the conditions for existence of a solution [3, 4] as well as solution strategies [5]. The generalized problem formulation based on choosing a gain matrix to achieve desirerable eigenvectors within the null space of

$$\begin{bmatrix} \lambda_j \mathbf{I} - \mathbf{A} & \mathbf{B} \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{v}_j \\ \boldsymbol{u}_j \end{bmatrix} = \begin{bmatrix} \mathbf{0} \end{bmatrix} \tag{1}$$

was probably first mentioned in [2] and is still used today. Herein $\mathbf{A} \in \mathbb{R}^n \times \mathbb{R}^n$ and $\mathbf{B} \in \mathbb{R}^n \times \mathbb{R}^m$ denote the system dynamics and input matrices, λ_j the eigenvalues and $\boldsymbol{v}_j$ the corresponding right-hand eigenvectors. Following the classification suggested by [6], four different types of solution approaches may be distinguished:

- *protection methods* being based on a multistep procedure, with one eigenvalue/-vector assigned in each step and protected from changes in the further steps by rendering the mode unobserv-

able/uncontrollable using a pre-compensator matrix, which reduces the output/input dimension by one in each step[7–9],

- *parametric methods* aiming on an explicitly parametrized representation of the subspace, where the eigenvectors are selectable from – this allows for an easy incorporation into outer procedures, which may optimize the eigenvectors to achieve higher level design goals [10],
- *orthogonal eigenvector methods* where the desired eigenvectors are not explicitly specified, but an orthogonal basis optimized for robustness and sensitivity issues is generated [5, 11, 12],
- *projection methods* which use the available degree of freedom in the eigenstructure to match a subset of the desired eigenvalues/eigenvectors either exactly or as best as possible in a least squares sense – this is achieved by augmenting Eq. 1 with additional equations relating the achievable eigenvectors $\boldsymbol{v}_j$ to the desired ones, thus reducing the nullspace and arriving at a uniquely solvable or even overdetermined system [2, 3, 13]. This is also the technique adopted for this work.

Since those times this powerful design technique has lost nothing of its appeal and has been used for various flight control applications. This includes early work dealing with decoupling [14], robustness [15, 16] and sensitivity analysis [17] as well as recent developments, focusing e.g. on application to unconventional configurations [18, 19] aeroelastic control of flexible aircraft [20] and robust control [21, 22].

Nevertheless, most of the published work deals with the problem of choosing righthand eigenvectors, focusing on full state feedback [4]. Extension to the output feedback case and sometimes the selection of output directions rather than the eigenvectors directly (output value shaping) is frequently addressed as well [2, 3, 10]. There are some approaches which also allow for the selection of left hand eigenvectors or a combination of both in a multistep procedure [7–10]. As the lefthand eigenvectors are related to the input space, this enables to design the amount of the various eigenmodes excitation caused by a specific input.

However, one aspect seldomly mentioned so far is, how to control the influence of a specific eigenmode on the actuation inputs. Due to the feedback of potentially all output variables to all system inputs by means of the gain matrix, the systems eigenmodes generally appear in all of the inputs, meaning that their typical frequency characteristics will be detectable in each input signal, as long as the mode is excited by any means. That holds even if the system is already decoupled with respect to the outputs. A possibility to control that influence could be of great interest, as it allows to shape the frequency content of the physical actuation variables, which might be subject to certain dynamic constraints (e.g. in common flight control applications, a rapid thrust change is to be avoided due to limited engine dynamics). On the other hand, this type of requirement to the eigenstructure cannot be easily translated into eigenvector properties, as the relation depends on the gain matrix itself, which is unknown a priori and to be determined in the procedure as well.

Roppenecker [23] probably first came up with concept of so called *invariant parameter vectors* ($\boldsymbol{u}_j$). He suggested to consider them being the true free design parameter of the eigenstructure assignment problem, rather than the eigenvectors themselves, because they are independent of the chosen state representation [24]. He recognized, that the parameter vectors indeed describe the modal composition of the actuation inputs. He showed further, that many of the known design procedures (among others also the output value shaping according to Moore [2]) may be deducted to just a special choice of these parameter vectors. He also developed several approaches to optimize the parameter vectors in order to achieve higher level goals, such as structural limitations of the gain matrix [23]. However, all these approaches have in common, that the desired parameter vectors are completely determined beforehand, and the gain matrix

$$\mathbf{K} = [\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n] \left[(\mathbf{A} - \lambda_1 \mathbf{I})^{-1} \mathbf{B} \boldsymbol{u}_1, \ldots, (\mathbf{A} - \lambda_n \mathbf{I})^{-1} \mathbf{B} \boldsymbol{u}_n \right]^{-1} \tag{2}$$

is directly calculated from the parameter vectors later. A mixed approach, where some components of the parameter vectors and some components of the eigenvectors or output directions are specified simultaneously, seems not considered by the authors.

This however has been described by Fichter and Stephan [25], who follow an approach to solve the homogenous equation system

$$\begin{bmatrix} \lambda_j \mathbf{I} - \mathbf{A} & \mathbf{B} \\ \mathbf{Z}_j & \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{v}_j \\ \boldsymbol{u}_j \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix} \tag{3}$$

where $\mathbf{Z}_j$ may comprise a set of $m - 1$ additional requirements (with m being the number of system inputs) regarding the supression of specific eigen modes in either states, output variables or the actuation inputs. Due to the homogeneous nature of the equation system, the approach is limited to the complete suppression of eigen modes.

We propose a very similar procedure, which combines the concept afore mentioned with the inhomogeneous formulation

$$\begin{bmatrix} \lambda_j \mathbf{I} - \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{0} \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{v}_j \\ \boldsymbol{u}_j \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \boldsymbol{y}_j^0 \end{bmatrix} \tag{4}$$

of the equation system as presented by Moore [2] and used also by Stevens et al. [13]. It involves one additional requirement compared to the homogeneous one (overall m requirements rather than $m - 1$). This assigns a concrete (but arbitrary) magnitude to the eigenvectors, which are uniquely defined only with respect to their direction. Therefore the approach offers the additional flexibility to specify also a (none zero) ratio, in which a certain eigenmode should contribute to the various quantities.

In extension to Moore [2] however, the proposed method has been generalized to allow for a flexible formulation of eigenstructure requirements with respect to both, output and/or input variables in an arbitrary combination.

The paper is structured as follows: In section 2 the eigenstructure assignment technique is reviewed as presented by Stevens et al. [13] to outline the foundation and introduce the nomenclature. Section 3 presents the suggested extensions and concludes with the generalized formulation. Finally, in section 4 the concept is applied to a linear decoupling controller of the longitudinal aircraft motion, to demonstrate the findings and their implications in praxis.

2 Review of Classical Eigenstructure Assignment (Projection Method)

Assuming a linear dynamic System

$$\dot{\boldsymbol{x}} = \mathbf{A}\boldsymbol{x} + \mathbf{B}\boldsymbol{u} \tag{5}$$

$$\boldsymbol{y} = \mathbf{C}\boldsymbol{x} \tag{6}$$

of n-th order, given in state space representation, the output feedback law

$$\boldsymbol{u} = -\mathbf{K}\boldsymbol{y} \tag{7}$$

may be applied. Further utilizing a prefilter

$$\boldsymbol{u} = \mathbf{L}\boldsymbol{w} \tag{8}$$

to map some reference commands $\boldsymbol{w}$ to the physical inputs $\boldsymbol{u}$ turns Eq. 5 to

$$\dot{\boldsymbol{x}} = \underbrace{(\mathbf{A} - \mathbf{BKC})}_{\widetilde{\mathbf{A}}}\boldsymbol{x} + \mathbf{BL}\boldsymbol{w} \tag{9}$$

with $\widetilde{\mathbf{A}}$ denoting the closed loop dynamic matrix. The dynamic behavior is characterized by its eigenvalues λ_j defined by the relationship

$$\lambda_j \boldsymbol{v}_j = \widetilde{\mathbf{A}}\boldsymbol{v}_j \Leftrightarrow (\lambda_j I - \mathbf{A} + \mathbf{BKC})\boldsymbol{v}_j = 0 \qquad (j \in \{1 \dots n\}) \tag{10}$$

where $\boldsymbol{v}_j$ is the (right-hand) eigenvector corresponding to λ_j. Following [13], Eq. 10 may be arranged in form of a linear system of equations

$$\begin{bmatrix} \lambda_j \mathbf{I} - \mathbf{A} & \mathbf{B} \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{v}_j \\ \boldsymbol{u}_j \end{bmatrix} = \begin{bmatrix} \mathbf{0} \end{bmatrix} \tag{11}$$

with

$$\boldsymbol{u}_j := \mathbf{KC}\boldsymbol{v}_j, \tag{12}$$

which contains the eigenvectors $\boldsymbol{v}_j$ and the instead of the gain matrix $\mathbf{K}$ newly introduced $\boldsymbol{u}_j$ as unknown variables. Obviously the solution is not unique – due to the ambiguity of possible eigenvectors assigned to each λ_j, any vector $\cdot \begin{bmatrix} \boldsymbol{v}_j & \boldsymbol{u}_j \end{bmatrix}^T$ lying within the nullspace of $\begin{bmatrix} \lambda_j \mathbf{I} - \mathbf{A} & \mathbf{B} \end{bmatrix}$ forms a valid solution. The System may be amended by additional equations specifying the desired system structure with respect to components of the eigenvectors $\boldsymbol{v}_j$, e.g.

$$\boldsymbol{v_j} := \boldsymbol{v}_j^0 \qquad (j \in \{1 \dots n\}) \tag{13}$$

which would yield

$$\begin{bmatrix} \lambda_j \mathbf{I} - \widetilde{\mathbf{A}} & \mathbf{B} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{v}_j \\ \boldsymbol{u}_j \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \boldsymbol{v}_j^0 \end{bmatrix} \tag{14}$$

As long as the number of inputs equals the system order ($m = n$), a unique solution will exist. If there are fewer inputs available, the system would be over determined - thus a decision has to be made, which of the additional equations could be waived. Solving the system for all k freely configurable eigenmodes and arranging the solution vectors in the matrices

$$\mathbf{V} = [\boldsymbol{v}_1 \quad \boldsymbol{v}_2 \dots \boldsymbol{v}_k] \qquad \text{and} \qquad \mathbf{U} = [\boldsymbol{u}_1 \quad \boldsymbol{u}_2 \dots \boldsymbol{u}_k] \tag{15}$$

finally allows to determine the gain matrix

$$\mathbf{K} = \mathbf{U}(\mathbf{CV})^{-1} \tag{16}$$

required to achieve the desired eigenstructure using Eq. 12.

However, the structure has been defined with respect to the eigenvectors themselves by specifying explicitly their components - that is the participation of the *state variables* on the respective eigenmode. This becomes clear in face of the systems modal representation: From Eq. 10 the dependency

$$\widetilde{\mathbf{A}} = \mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^{-1} \tag{17}$$

may be derived, with

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \tag{18}$$

being the diagonal matrix of closed loop eigenvalues. Substitution in Eq. 5 yields the modal form

$$\dot{\boldsymbol{x}}_d = \mathbf{\Lambda}\boldsymbol{x}_d + \mathbf{B}_d\boldsymbol{u} \tag{19}$$

$$\boldsymbol{y} = \mathbf{C}_d\boldsymbol{x}_d + \mathbf{D}\boldsymbol{u} \tag{20}$$

where

$$\boldsymbol{x}_d = \mathbf{V}^{-1}\boldsymbol{x} \qquad \mathbf{B}_d = \mathbf{V}^{-1}\mathbf{B} \qquad \text{and} \qquad \mathbf{C}_d = \mathbf{C}\mathbf{V} \tag{21}$$

are the modal state vector, input and output matrix. Alternatively, the dual representation based on the left hand eigenvectors taken as row vectors $\boldsymbol{w}$ satisfying

$$\lambda_j \boldsymbol{w}_j = \boldsymbol{w}_j \widetilde{\mathbf{A}} \tag{22}$$

may be employed, yielding

$$\widetilde{\mathbf{A}} = \mathbf{W}^{-1}\mathbf{\Lambda}\mathbf{W} \tag{23}$$

with

$$\mathbf{W} = \begin{bmatrix} \boldsymbol{w}_1 \\ \boldsymbol{w}_2 \\ \vdots \\ \boldsymbol{w}_k \end{bmatrix} \tag{24}$$

being the matrix of left hand eigenvectors. This yields equally well to the modal form 19 of this system, using this time

$$\boldsymbol{x}_d = \mathbf{W}\boldsymbol{x} \qquad \mathbf{B}_d = \mathbf{W}\mathbf{B} \qquad \text{and} \qquad \mathbf{C}_d = \mathbf{C}\mathbf{W}^{-1}. \tag{25}$$

From Eq. 21 follows directly $\boldsymbol{x} = \mathbf{V}\boldsymbol{x}_d$, spelling out that $\mathbf{V}$ – and thus the specified $\boldsymbol{v}_j^0$ – indeed describe the contribution of each modal amplitude to the original state variables. Forcing e.g. $\mathbf{V} = \mathbf{I}$ by choosing $v_{ij}^0 = \delta_{ij}$ would yield a closed loop system which is fully decoupled with respect to the chosen state representation $\boldsymbol{x}$, because each state variable participates on one single eigenmode only. The output variables $\boldsymbol{y}$ however remain still influenced by multiple eigenmodes, as they are composed of different system states by means of the $\mathbf{C}$ matrix.

For most applications however, the goal would be to specify the dynamics with respect to the system outputs rather than the internal state representation, which translates to demand

$$\boldsymbol{y} = \mathbf{Y}^0\boldsymbol{x}_d \tag{26}$$

where

$$\mathbf{Y}^0 = \begin{bmatrix} \boldsymbol{y}_1^0 & \boldsymbol{y}_2^0 \ldots \boldsymbol{y}_k^0 \end{bmatrix} \tag{27}$$

denote the *output directions*, that is, the participation of each output variable on the jth eigenmode. This may obviously be achieved by selecting

$$\mathbf{V}^0 = \mathbf{C}^{-1}\mathbf{Y}^0 \tag{28}$$

However, the explicit calculation will fail for any none square systems where the number of inputs or outputs does not match the system order. The problem may be circumvented, when Eq.29 is integrated

[13] in the system 11 to solve, yielding

$$\begin{bmatrix} \lambda_j \mathbf{I} - \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{0} \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{v}_j \\ \boldsymbol{u}_j \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \boldsymbol{y}_j^0 \end{bmatrix} \tag{29}$$

This approach still works out for non square **C**-matrices ($k < n$) as well as for the case of insufficient number of inputs ($m < n$), where several equations have to be omitted to end up with a solvable system.

3 Extension to Specification of Feedback Directions

The eigenstructure assignment method, as presented so far, allows to define the desired structure not only in terms of the eigenvectors themself (that is, with respect to the internal system states), but also in terms of output directions. However, considering the system inputs

$$\boldsymbol{u} = -\mathbf{K}\boldsymbol{y} = -\mathbf{K}\mathbf{Y}^0\boldsymbol{x}_d \tag{30}$$

we find that those are still influenced by various eigenmodes in an unspecified way, because the gain matrix in general is not diagonal. Therefore, many output variables contribute to each input, and even if the outputs are decoupled, the inputs are not.

Assuming the goal would be to decouple the system inputs, or, more generally, to specify, which influence a certain eigenmode should have on a specific input, e.g.

$$\boldsymbol{u} := \mathbf{U}^0\boldsymbol{x}_d \tag{31}$$

with

$$\mathbf{U}^0 = \begin{bmatrix} \boldsymbol{u}_1^0 & \boldsymbol{u}_2^0 \dots \boldsymbol{u}_n^0 \end{bmatrix} \tag{32}$$

to be called the matrix of required *feedback directions* within the scope of this paper. Clearly, that could be achieved by selecting

$$\mathbf{V}^0 = (\mathbf{KC})^{-1}\,\mathbf{U}^0, \tag{33}$$

but unless the systems has as many independent inputs as it has states, **KC** is not invertable. Further more, the resulting gain matrix **K** is unknown prior to solution of Eq. 14, which on itself depends on $\mathbf{V}^0$. Again, this problem may be avoided by incorporating the relationship 32 into the system. Thanks to the definition 12, this is easily accomplished by forcing $\boldsymbol{u}_j = \boldsymbol{u}_j^0$, yielding

$$\begin{bmatrix} \lambda_j \mathbf{I} - \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{v}_j \\ \boldsymbol{u}_j \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \boldsymbol{u}_j^0 \end{bmatrix} \tag{34}$$

With Eq. 29 and 34 we have on hand now the tools required to specify the system structure in terms of either the outputs, or the inputs participation on eigenmodes. However, it is impossible to achieve both for all in- and outputs simultanously. The system of equations has a number of $n + m$ unknown variables ($\boldsymbol{v}_{j1} \dots \boldsymbol{v}_{jn}$ and $\boldsymbol{u}_{j1} \dots \boldsymbol{u}_{jm}$). The eigenvalue equation (upper rows) makes up n constraints, leaving over an amount of m degrees of freedom. These need to be specified by m additional equations defining the system structure, which may address the eigenmode participation of outputs and/or inputs, but both together representing exactly m requirements.

Therefore a selection has to be made, for which of the inputs and outputs requirements shall be specified, or in other words, which of the additional k equations in Eq. 29 or m in Eq. 34 to waive on.

This may be accomplished using a boolean selection matrix

$$\mathbf{S}_j = \begin{bmatrix} \mathbf{S}_y & 0 \\ 0 & \mathbf{S}_u \end{bmatrix}, \quad \mathbf{S}_j \in \mathbb{B}^m \times \mathbb{B}^{n+m},$$

which yields with the definitions

$$\mathbf{D}_j = \mathbf{S}_j \cdot \begin{bmatrix} \mathbf{C} & 0 \\ 0 & \mathbf{I} \end{bmatrix} \qquad \text{und} \qquad \boldsymbol{r}_j = \mathbf{S}_j \cdot \begin{bmatrix} \boldsymbol{y}_j^0 \\ \boldsymbol{u}_j^0 \end{bmatrix} \tag{35}$$

the system of equations

$$\begin{bmatrix} \lambda_j \mathbf{I} - \mathbf{A} & \mathbf{B} \\ \mathbf{D}_j & \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{v}_j \\ \boldsymbol{u}_j \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \boldsymbol{r}_j \end{bmatrix} \tag{36}$$

to solve. The selection matrix determines, for which of the outputs and inputs requirements addressing their participation on the j-th eigenmode shall be formulated. The overall number of requirements has to equal the number of system inputs (m), but may be arbitrarily distributed between outputs (defined by $\mathbf{S}_y$) and inputs (defined by $\mathbf{S}_u$).

It should be noted that the presented method allows to decouple the inputs in the sense that the feedback applied to a specific input represents the *influence* of a single eigenmode solely. This has not to be confused with the question, whether a certain input *excites* only a single eigenmode. In order to control the influence of an input on the systems eigenmodes, one would like to specify the *input directions* $\mathbf{Z}^0$ in

$$\dot{\boldsymbol{x}}_d := \boldsymbol{\Lambda} \boldsymbol{x}_d + \mathbf{Z}^0 \boldsymbol{u}. \tag{37}$$

This is, following Eq. 21, achieved by choosing

$$\mathbf{V}^0 = \mathbf{B} \left(\mathbf{Z}^0 \right)^{-1}, \tag{38}$$

or alternatively, using the dual equations 25, by selecting

$$\mathbf{W}^0 = \left(\mathbf{Z}^0 \right) \mathbf{B}^{-1}. \tag{39}$$

However, the direct inversion is limited to square ($m = n$) systems.

Decoupling the inputs in the latter way – that is letting each eigenmode to be excited by a single input only – might also be of less practical relevance, as for any given eigenstructure $\mathbf{V}$, a static prefilter

$$\mathbf{L} = - \left[\mathbf{C}_r (\mathbf{A} - \mathbf{BKC})^{-1} \mathbf{B} \right]^{-1} \tag{40}$$

may be designed [13], which ensures that the new reference signals $\boldsymbol{w}$ used to calculate the input

$$\boldsymbol{u} = \mathbf{L} \boldsymbol{w} \tag{41}$$

represent the stationary end values of arbitrary outputs

$$\boldsymbol{y}_r = \mathbf{C}_r \boldsymbol{x}. \tag{42}$$

As long as the system is decoupled beforehand with respect to $\boldsymbol{y}_r$, each reference signal will also affect a single eigenmode only.

4 Application Example

The procedure shall be applied for control of the longitudinal aircraft motion. Assume a linearized model given in the state space representation

$$\begin{aligned} \dot{\boldsymbol{x}} &= \mathbf{A}\boldsymbol{x} + \mathbf{B}\boldsymbol{u} \\ \boldsymbol{y} &= \mathbf{C}\boldsymbol{x} + \mathbf{D}\boldsymbol{u} \end{aligned}$$

with

$$\boldsymbol{u} = \begin{bmatrix} \eta \\ \kappa \\ F \end{bmatrix} \qquad \boldsymbol{x} = \begin{bmatrix} \alpha \\ q \\ h \\ \gamma \\ V \end{bmatrix} \qquad \boldsymbol{y} = \begin{bmatrix} \Theta \\ q \\ h \\ \dot{h} \\ V \end{bmatrix}$$

being the input, state and output vectors. The deflections of elevator η and direct lift control flaps κ (DLC-flaps) as well as the thrust force F are available as inputs. The system state is represented by angle of attack α, pitch rate q, altitude h, flight path angle γ and the flight speed V. Pitch angle Θ and rate, altitude, vertical speed $\dot{h}$ and the flight speed are used as output and feedback variables. The numerical values for the dynamic matrix A, input matrix B, output matrix C and feedthrough matrix D are given in the appendix.

The open loop system shows the well known eigenmodes

short period with $\lambda_{1,2} = -3.34 \pm 1.39\mathrm{i}\,\mathrm{rad\,s^{-1}}$,
phugoid with $\lambda_{3,4} = -0.0178 \pm 1.07\mathrm{i}\,\mathrm{rad\,s^{-1}}$,
free altitude integration with $\lambda_5 = 0\,\mathrm{rad\,s^{-1}}$.

Assuming the goal is to decouple the vertical (h) and the longitudinal (V) degree of freedom from the pitch motion (Θ), the closed loop system is expected to modify the natural system dynamics in a way, that the following eigenmodes do appear instead:

pitch mode with $\lambda_{1,2} = -3 \pm 3\mathrm{i}\,\mathrm{rad\,s^{-1}}$,
plunge mode with $\lambda_{3,4} = -1 \pm \mathrm{i}\,\mathrm{rad\,s^{-1}}$,
speed dynamics with $\lambda_5 = -0.1\,\mathrm{rad\,s^{-1}}$.

The eigenvalues have been chosen more or less arbitrarily, assuming that a rather stiff control is desired to accurately track high dynamical attitude and flight path profiles, whereas the flight speed is expected to change significantly slower and higher deviations are tolerated. Furthermore, high frequent changes in the thrust input are to be explicitly avoided due to the engine characteristics. This leads directly to the requirement, that the thrust force feedback should only participate on the speed dynamics mode, and not on the high frequent pitch and plunge modes. To achieve decoupling of the pitch and plunge motion, it is further required, that the pitch angle will not participate on the plunge mode, and that the altitude will not be influenced by the pitch mode. This compiles to the following requirements scheme for the output

and feedback directions, where unspecified components are marked as $\star$.

$$\begin{bmatrix} \mathbf{Y0} \\ \mathbf{U0} \end{bmatrix} = \begin{array}{c} \\ \Theta \\ q \\ h \\ \dot{h} \\ V \\ \eta \\ \kappa \\ F \end{array} \begin{array}{c} \begin{array}{ccc} \lambda_{1,2} & \lambda_{3,4} & \lambda_5 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ \star & \star & \star \\ 0 & 1 & 0 \\ \star & \star & \star \\ \star & \star & \star \\ \star & \star & \star \\ \star & \star & \star \\ 0 & 0 & 1 \end{pmatrix} \end{array} \tag{43}$$

Note that one of the output or feedback direction components has to differ from zero, in order to define their scaling. (Any arbitrary variable which is independent of the ones set to zero already may be choosen here, e.g. also a '1' or any other non-zero value in the row correspondig to V instead of F, however we decided to complete the columns, where the zeros have been specified already.) Furthermore, as the oscillatory modes are represented by complex conjugate eigenvalues, the eigenvectors and thus also output/feedback directions shall also be complex conjugate, e.g. identical. Thats why they have been summarized in Eq. 43 above.

From Eq. 43 the Selection matrices

$$\mathbf{S}_{1,2} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{S}_{3,4} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$
$$\mathbf{S}_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{44}$$

may be determined by deleting from a $m \times k$ identity matrix those rows, which don't define any requirement in $\left[\mathbf{Y}^0\mathbf{U}^0\right]^T$ ($\star$ elements) in the column for the considered eigenvalue.

Using Eq. 35 and solving 36 for all eigenvalues, the actual feedback directions

$$\mathbf{U} = [\boldsymbol{u}_1 \quad \boldsymbol{u}_2 \ldots \boldsymbol{u}_5] \qquad \text{and eigenvectors} \qquad \mathbf{V} = [\boldsymbol{v}_1 \quad \boldsymbol{v}_2 \ldots \boldsymbol{v}_5] \tag{45}$$

may be determined. Eq. 16 finally allows to solve for the required gain matrix $\mathbf{K}$ and the prefilter $\mathbf{L}$ may be calculated using Eq. 40. Both are given in the appendix.

Fig. 1 shows the step response of the system to a pitch angle command, Fig. 2 for an altitude, and Fig. 3 for a speed command. In all cases the actual values follow the reference without stationary error, which proves the correct prefilter design. Due to the prefilter structure, the pitch angle command excites only pitch mode ($\lambda_{1,2} = 3 + 3\mathrm{i}\,\mathrm{rad\,s^{-1}}$) and speed dynamics ($\lambda_5 = 0.1\,\mathrm{rad\,s^{-1}}$), the plunge mode remains inactive. In consequence, pitch rate and pitch angle show a decent reaction, whereas the deviations of vertical speed and altitude from their reference values stay below the numerical accuracy. In contrast to a full decoupling of system outputs, here the speed dynamics is still active and it becomes clear from the velocity response that both, the slow speed dynamics and the fast pitch mode exert their influence on the velocity. In contrast, the thrust input shows only the slow, low frequent response attributed to the speed dynamics. (The thrust required for stationary flight decreases, because the DLC-flap deflection is reduced compared to the reference state, in order to compensate for the increased angle of attack. This results in a cleaner configuration with lower drag.) The pitch mode doesn't take any effect here, as enforced by

the design. As no decoupling has been forced for the elevator and DLC-flap inputs, in principle they are affected by all the dynamic modes. However, because the plunge mode is not excited by pitch commands at all, only the effect of the slightly higher frequent pitch mode ($\lambda_{1,2} = 3 + 3\mathrm{i}\,\mathrm{rad\,s}^{-1}$) can be observed in those inputs.

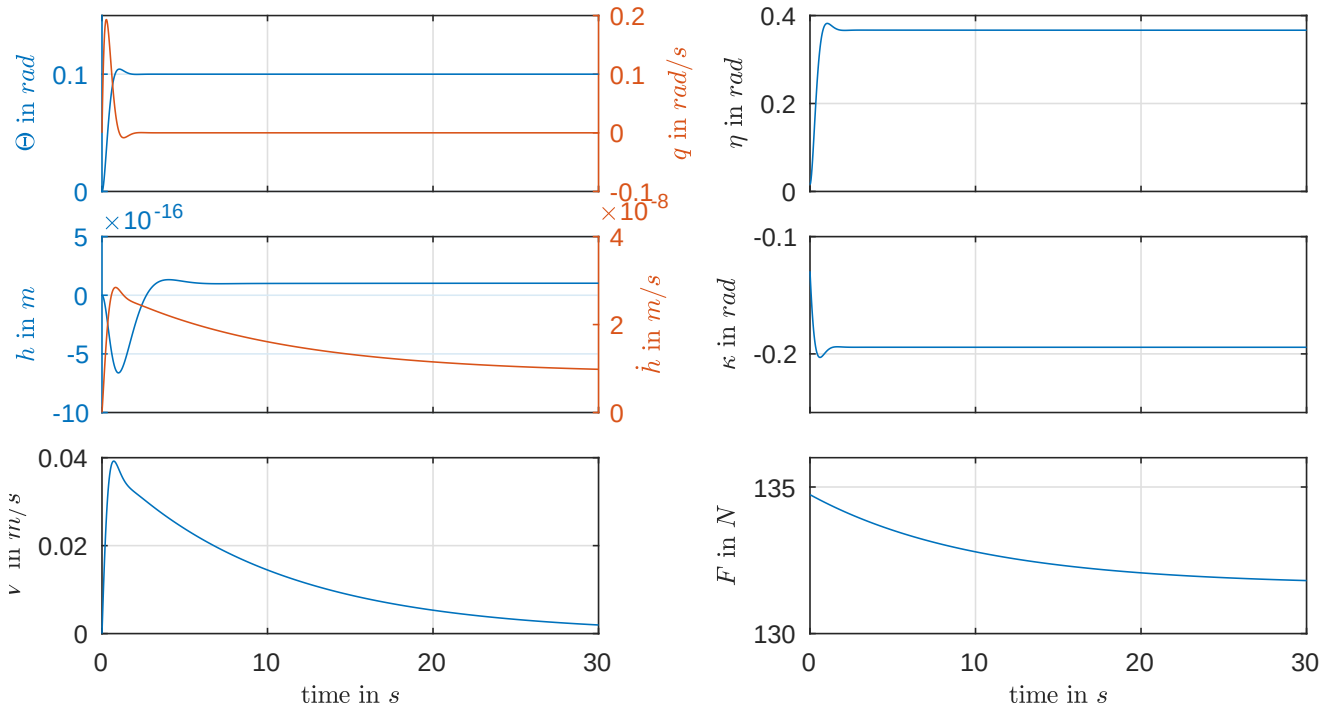

Fig. 1 Time response for an unit step in pitch angle command

The response to an altitude unit step command (Fig. 2) draws the opposite picture: Here the pitch mode remains unexcited, whereas plunge mode and speed dynamics are active. Consequently, the pitch angle shows no reaction, as it it is solely influenced by the pitch mode. The plunge mode ($\lambda_{3,4} = 1 + \mathrm{i}\,\mathrm{rad\,s}^{-1}$) exerts its influence on altitude and vertical speed, which is not the case for the lower frequent speed dynamics. The latter however affects both, velocity and thrust input. It can be observed again, that the velocity participates also on the plunge mode, whereas the thrust input does not.

For the velocity step command (Fig. 3), the prefilter guarantees, that the speed dynamics is the only mode being excited. This results in the absence of any reaction in both, the pitch (Θ and q) as well as the plunge (h and $\dot{h}$) degree of freedom, because neither of it shall be influenced by the speed mode according to Eq.43. The velocity and all the inputs however show the slow reaction characteristics for the speed dynamics, but no higher frequent content at all, which is because neither the pitch nor the plunge mode has been excited by the command.

It should be noted, that the unusual configuration featuring DLC-flaps has been explicitly chosen here to demonstrate the method proposed in section 3 and illustrate the difference between decoupling of output and feedback directions. As already pointed out in section 3, the number of degree of freedom available to shape the system structure beyond the pure eigenvalue placement matches the number of system inputs m. Assuming there would be no DLC-flaps available, the eigenstructure specification would be limited to choosing only two (rather than three) components in each column of the requirements scheme 43. That would render a simultaneous decoupling of pitch and plunge mode impossible whilst excluding those modes from the thrust input. Only one goal could be achieved at a time. The advantage to consider DLC-flaps however is, that there are as many inputs available as there are eigenmodes (pitch mode, plunge mode, speed dynamics) involved. This would theoretically enable a full decoupling as well, which contrasts nicely to the approach demonstrated here.

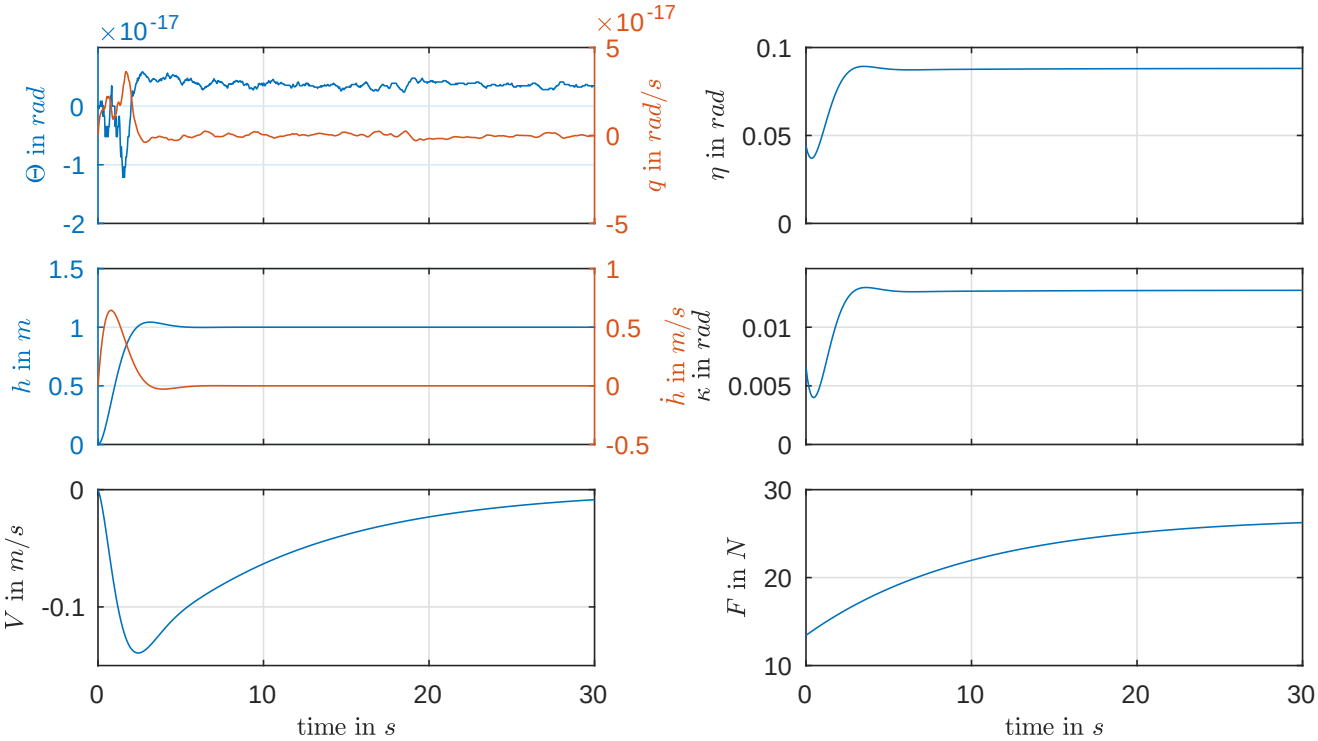

Fig. 2 Time response for an unit step in altitude command

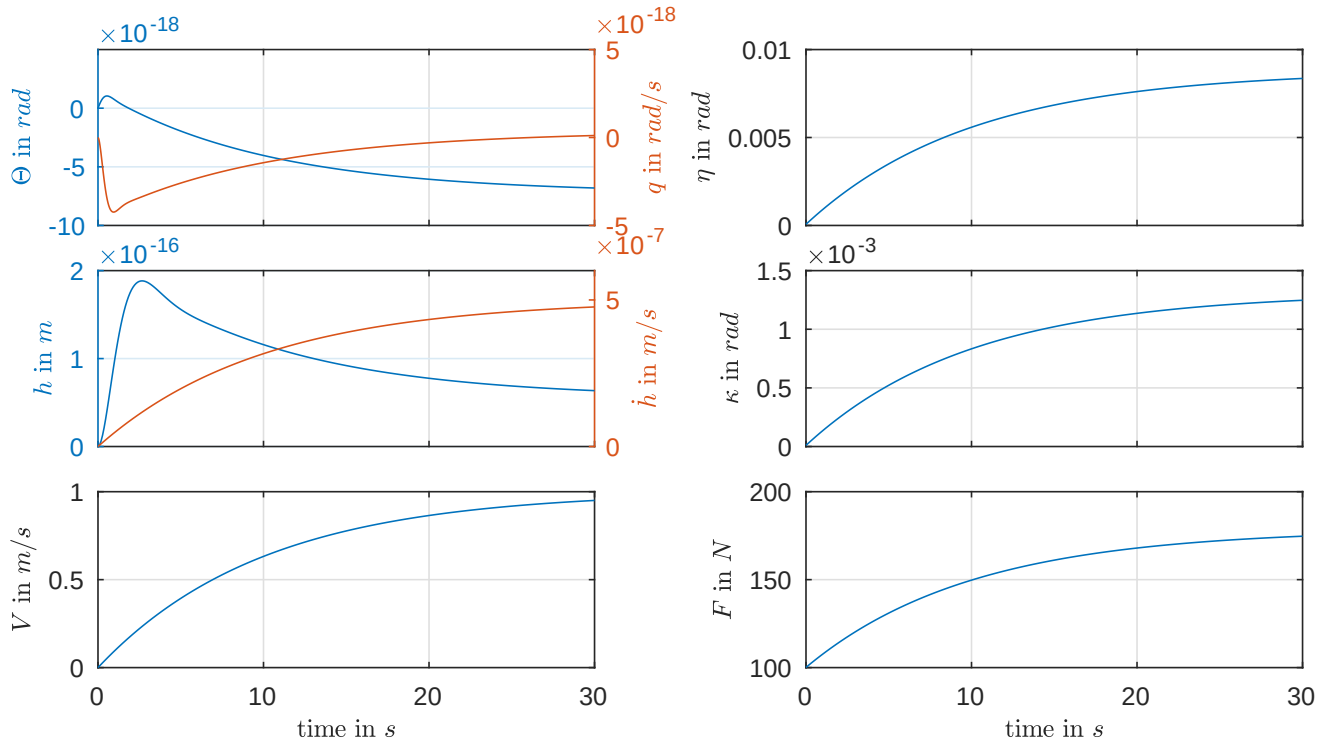

Fig. 3 Time response for an unit step in flight speed command

5 Conclusion

An extension to the eigenstructure assignment method has been proposed, which allows to decouple or specify the desired system structure not only in terms of eigenvectors or output directions, but also with respect to the control inputs of the physical plant. This enables to specify the desired participation of certain control inputs on the various system eigenmodes. This can be especially useful, if some of the physical actuating variables shall not be employed in certain control channels/control error constellations, or if their nature requires a restricted frequency content. Furthermore, a consistent design methodology has been derived, suporting also the combination of different eigenstructure requirements formulated in terms of both, output- and input variables. This method has been demonstrated by designing a linear decoupling controller for the longitudinal aircraft motion, which prevents the high frequent eigenmodes from affecting the thrust input.

Appendix

All coefficients are given in the appropriate SI-units corresponding to the state-, input- or output variables they refer to.

$$\mathbf{A} = \begin{bmatrix} -3.05 & 1.01 & 1.89e-05 & 0 & -0.00787 \\ -1.99 & -3.63 & 0 & 0 & -2.07e-06 \\ 0 & 0 & 0 & 50 & -5.01e-07 \\ 3.06 & 4.84e-05 & -1.89e-05 & 0 & 0.00787 \\ -4.98 & -0.0394 & 7.44e-05 & -9.81 & -0.0309 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} -0.0513 & -0.9 & 6.15e-07 \\ -13.7 & 2.04 & 0 \\ 0 & 0 & 0 \\ 0.106 & 0.891 & -6.15e-07 \\ 0 & -1.07 & 0.001 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 50 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$
\mathbf{K} = \begin{bmatrix} -0.649 & -0.175 & 0.00658 & -0.00635 & 0.0013 \\ 3.51 & 0.0209 & 0.0441 & -0.0231 & 0.00872 \\ -30.8 & -5.44 & 13.5 & 2.01 & 78.5 \end{bmatrix}
$$

$$
\mathbf{L} = \begin{bmatrix} -1.29 & 0.00658 & 1.01e-05 \\ 0.155 & 0.0441 & 6.78e-05 \\ 1.35e+03 & 13.4 & 100 \end{bmatrix}
$$

References

[1] Hidenori Kimura. Pole assignment by gain output feedback. *IEEE Transactions on automatic Control*, 20(4):509–516, 1975.

[2] Bruce C Moore. On the flexibility offered by state feedback in multivariable systems beyond closed loop eigenvalue assignment. In *1975 IEEE Conference on Decision and Control including the 14th Symposium on Adaptive Processes*, pages 207–214. IEEE, 1975.

[3] S Srinathkumar. Eigenvalue/eigenvector assignment using output feedback. *IEEE Transactions on Automatic Control*, 23(1):79–81, 1978.

[4] V Sinswat and F Fallside. Eigenvalue/eigenvector assignment by state-feedback. *International Journal of Control*, 26(3):389–403, 1977.

[5] A Bradshaw, LR Fletcher, and B Porter. Synthesis of output-feedback control laws for linear multivariable continuous-time systems. *International Journal of Systems Science*, 9(12):1331–1340, 1978.

[6] BA White. Eigenstructure assignment: a survey. *Proceedings of the Institution of Mechanical Engineers, Part I: Journal of Systems and Control Engineering*, 209(1):1–11, 1995.

[7] LR Fletcher and Jean Francis Magni. Exact pole assignment by output feedback part 1. *International Journal of Control*, 45(6):1995–2007, 1987.

[8] LR Fletcher. Exact pole assignment by output feedback part 2. *International Journal of Control*, 45(6):2009–2019, 1987.

[9] LR Fletcher. Exact pole assignment by output feedback part 3. *International Journal of Control*, 45(6):2009–2019, 1987.

[10] MM Fahmy and J O'REILLY. Multistage parametric eigenstructure assignment by output-feedback control. *International Journal of Control*, 48(1):97–116, 1988.

[11] SK Mudge and RJ Patton. Analysis of the technique of robust eigenstructure assignment with application to aircraft control. In *IEE Proceedings D-Control Theory and Applications*, volume 135, pages 275–281. IET, 1988.

[12] J Kautsky, NK Nichols, Paul Van Dooren, and L Fletcher. Numerical methods for robust eigenstructure assignment in control system design. In *Numerical Treatment of Inverse Problems in Differential and Integral Equations*, pages 171–178. Springer, 1983.

[13] B.L. Stevens, F.L. Lewis, and E.N. Johnson. Modern design techniques. In *Aircraft Control and Simulation: Dynamics, Controls Design, and Autonomous Systems*, pages 377–499. Wiley, 2015. ISBN: 9781118870983.

[14] Kenneth M Sobel and Elizer Y Shapiro. A design methodology for pitch pointing flight control systems. *Journal of Guidance, Control, and Dynamics*, 8(2):181–187, 1985.

[15] Kenneth M Sobel and Frederick J Lallman. Eigenstructure assignment for the control of highly augmented aircraft. *Journal of Guidance, Control, and Dynamics*, 12(3):318–324, 1989.

[16] KENNETH M SOBEL, WANGLING YU, JEAN E PIOU, JAMES R CLOUTIER, and ROBERT WILSON. Robust eigenstructure assignment with structured state-space uncertainty and unmodelled dynamics. *International journal of systems science*, 23(5):765–788, 1992.

[17] A NAYAK and H YOUSSEF. Sensitivity analysis of generalized eigenstructure assignment using constant output feedback. In *Guidance, Navigation and Control Conference*, page 4102, 1988.

[18] Clara Nieto-Wire and Kenneth Sobel. Eigenstructure assignment for a tailless aircraft. In *AIAA Guidance, Navigation and Control Conference and Exhibit*, page 6417, 2007.

[19] Alexander Köthe and Robert Luckner. Applying eigenstructure assignment to inner-loop flight control laws for a multibody aircraft. *CEAS Aeronautical Journal*, pages 1–11, 2021.

[20] Rafael M Bertolin, Antônio B Guimarães Neto, Guilherme C Barbosa, Juliano A Paulino, and Flávio J Silvestre. Design of stability augmentation systems for flexible aircraft using projective control. *Journal of Guidance, Control, and Dynamics*, 44(12):2244–2262, 2021.

[21] Christoph Thaddäus Baier. *Entwicklung und Verifizierung eines Konzepts zur Robusten Regelung der Seitenbewegung zukünftiger Kleinflugzeuge*. PhD thesis, Technische Universität München, 2017.

[22] Onur Albostan and Metin Gökaşan. Mode decoupling robust eigenstructure assignment applied to the lateral-directional dynamics of the f-16 aircraft. *Aerospace Science and Technology*, 77:677–687, 2018.

[23] G Roppenecker. Vollstandige modale synthese linearer systeme und ihre anwendung zum entwurf strukturbeschr˜ inkter zustandsriickfiihrungen. *Fortschritt-Berichte der VDl-Zeitschriften*, 1983.

[24] G Roppenecker. Pole assignment by state feedback. *Regelungstechnik*, 29:228–233, 1981.

[25] Walter Fichter and Johannes Stephan. *Flugregelung: Theoretische Grundlagen für die Lenkung und Regelung von Flächenflugzeugen*. Springer-Verlag, 2020.