CEAS-GNC-2022-052

CEAS EuroGNC 2022
"Conference on Guidance, Navigation and Control"
3-5 May 2022 @ Technische Universität Berlin, Germany

Local Stability Analysis for Sensor-based Inexact Feedback Linearization

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ABSTRACT

Sensor-based feedback laws such as incremental nonlinear dynamic inversion (INDI) applied to flight control tasks have been successfully evaluated in experiments. When deriving sensor-based feedback laws by INDI, it is assumed that a state-dependent error term introduced by Taylor approximation of the exact feedback linearization is neglectable. In reality, this assumption does not hold over the full state-space and closed-loop stability is a local property. When written in the co-space of internal and external dynamics obtained by feedback linearization, a local characterization for the state-dependent error term by a finite-horizon output gain is proposed. Thus, an inner estimate of the closed-loop region of attraction is derived by application of the small-gain theorem to the interconnection of internal and external dynamics as well as the over-approximated error term.

Keywords: Incremental nonlinear dynamic inversion, Region of attraction, Input-to-state stability, Small-gain theorem, Finite-horizon output gain

Nomenclature

$x, \mathbb{X}$: states and state-space ($x \in \mathbb{X}$)
$s, \mathbb{S}$: co-states and co-space ($s \in \mathbb{S}$)
$u, \mathbb{U}$: inputs and input space ($u \in \mathbb{U}$)
$v, \mathbb{V}$: pseudo-input and space ($v \in \mathbb{V}$)
$y, \mathbb{Y}$: outputs and output space ($y \in \mathbb{Y}$)
$\mathscr{K}_\mathbb{U}$: set of linear mappings $K : \mathbb{U} \to \mathbb{X}$
f : nonlinear state dynamics ($f : \mathbb{X} \to \mathbb{X}$)
g : nonlinear input dynamics ($g : \mathbb{X} \to \mathscr{K}_\mathbb{U}$)
h : nonlinear output dynamics ($h : \mathbb{X} \to \mathbb{Y}$)

1 Introduction

Exact feedback linearization, also known as *nonlinear dynamic inversion* (NDI), is a classical method for the derivation of linearizing control laws for nonlinear dynamics [1]. While NDI has been applied to flight control [2–5], it depends on an algebraic model in the inversion. On the other hand, reliable control-oriented flight dynamics models are often not available to ensure robust control through NDI. In order to increase robustness to modeling uncertainties, *incremental nonlinear dynamic inversion* (INDI) has been proposed [6–8]. Here, an algebraic model is partially replaced by sensor data – in particular, measured accelerations – and these measurements inform a first-order approximation of the nonlinear dynamics. While initially proposed for attitude control of fixed-wing aircraft, INDI has since been applied for guidance and tracking tasks for various configuration including rotary-wing and transitioning aircraft [8–13]. Although INDI has been demonstrated in numerous fight experiments, it lacks a rigorous foundation in control theory. The use of an inexact feedback linearization at the core of INDI is often dismissed by the assumption that the dynamics of the controls are significantly faster than the change of the states (*timescale separation*). It is usually under this paradigm that stability of INDI as well as its robustness to model uncertainties and disturbances is derived [8, 13–15]. A notable exception is the work of [16], in which the error term introduced by the first-order approximation is described and investigated.

We note that this error term, since state-dependent, might be locally neglectable but the assumption of timescale separation is unlikely to hold for all states. In this paper, we therefore propose a method for estimating the domain of stability or *region of attraction*. To that extent, we make use of the co-space of internal and external dynamics, in which the interdependency of the error term is exposed [16]. While the external dynamics are simply an higher-order integrator, the internal dynamics, since hidden, are often required to be robustly stable [1, 17]. In the case of exact feedback linearization, with properly chosen linear feedback matrix, this requirement leads to asymptotic stability of the closed-loop dynamics. Such an immediate result is unavailable for the inexact linearization because of the state-dependent error term. We now introduce and locally over-approximate the error by a nonlinear finite-horizon output gain based on [18] and verify local stability by application of the small-gain theorem. In this paper, we focus on the error term induced by the inexact linearization, whereas model uncertainties are left to future work.

The remainder of the paper is organized as follows: We introduce exact and inexact linearization, notions of stability, and finite-horizon output gains in Section 2. As the main result, a region of local stability is derived in Section 3, where the error term is replaced by its output gain. In Section 4 then, methods for the computation of the finite-horizon output gain are discussed. Lastly, the proposed methodology is illustrated on a numerical example in Section 5.

Notation

Let $\mathbb{X}$ and $\mathbb{S}$ be finite-dimensional vector spaces. Given a vector $x = (x_1, \dots, x_n) \in \mathbb{X}$, the $\mathscr{L}_p$-norm of x is $||x||_p = \sqrt[p]{|x_1|^p + \cdots + |x_n|^p}$. The supremum norm of $v : [0,\infty) \to \mathbb{X}$ is $||v||_\infty = \sup_{t\geq 0} |v(t)|$. The ball of radius $r > 0$ in $\mathbb{S}$ is denoted by $\mathscr{B}_r(\mathbb{S})$ and defined as the set of all $s \in \mathbb{S}$ satisfying $|s| \leq r$. A smooth, strictly increasing function $\alpha : [0,\infty) \to [0,\infty)$ with $\alpha(0) = 0$ is said to be of class $\mathscr{K}$; and $\mathscr{KL}$ is the class of all smooth functions $\beta : [0,\infty) \times [0,\infty) \to [0,\infty)$ where $\beta(\cdot,t) \in \mathscr{K}$ if $t \geq 0$ is fixed and $\beta(r,\cdot)$ is strictly decreasing with $\lim_{t\to\infty} \beta(r,t) = 0$ for any $r \geq 0$.

2 Problem Statement

We consider a system given by the nonlinear differential equations in control-affine form,

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t) \tag{1a}$$

$$y(t) = h(x(t)) \tag{1b}$$

for almost all $t \geq 0$ with states $x(t) \in \mathbb{X}$, inputs $u(t) \in \mathbb{U}$, and outputs $y \in \mathbb{Y}$. We assume that $\mathbb{U}$ and $\mathbb{Y}$ both have dimension one.[1] The functions f, g, and h are assumed to be differentiable with

[1] Note that multi-input multi-output can be treated in the same framework [16], but are omitted for simplicity.

continuous i-th derivative for $i \in [1,n)$. Assume without loss of generality that $f(0)=0$, $h(0)=0$, and $g(0)\neq 0$.

2.1 Exact linearization

Differentiating the output by time, under application of the chain-rule, for $\rho < n$ times yields

$$y^{(\rho)}(t) = a(x(t)) + b(x(t))u(t) \tag{2}$$

for almost all $t \geq 0$, where $a : \mathbb{X} \to \mathbb{Y}$ and $b : \mathbb{X} \to \mathscr{K}_{\mathbb{Y}}^{*}$ are the input-output dynamics. Introducing the diffeomorphism $T : \mathbb{X} \to \mathbb{S}$, where $\mathbb{S} = \mathbb{S}_1 \times \mathbb{S}_2$, we obtain the co-dynamics

$$\dot{s}_1(t) = \phi(s_1(t), s_2(t)) \tag{3a}$$
$$\dot{s}_2(t) = A_{\mathrm{i}} s_2(t) + B_{\mathrm{i}} v(t) \tag{3b}$$
$$y(t) = C_{\mathrm{i}} s_2(t) \tag{3c}$$

for almost all $t \geq 0$ where $\phi : \mathbb{S}_1 \times \mathbb{S}_2 \to \mathbb{S}_1$ are the *internal* co-dynamics and $A_{\mathrm{i}} : \mathbb{S}_2 \to \mathbb{S}_2$, $B_{\mathrm{i}} : \mathbb{V} \to \mathbb{S}_2$, $C_{\mathrm{i}} : \mathbb{S}_2 \to \mathbb{Y}$ is a continuous integrator of order ρ (the *external* co-dynamics). Eq. (3) with pseudo-input

$$v(t) \equiv a(x(t)) + b(x(t))u(t) \tag{4}$$

is an exact linearization of (1) in the sense that x is a solution of (1) for u if and only if $s = T \circ x$ satisfies (3) for v. Assuming that $b(x) \neq 0$ for all $x \in \mathbb{X}$, we can design a feedback law $v = Ks_2$, where $K : \mathbb{S}_2 \to \mathbb{V}$ is a linear function, for the system in (1). The dynamics in (3a) for $s_2 \equiv 0$ are denoted ϕ_0 and called *zero dynamics*. Assume, again without loss of generality, that $T(0) = 0$ and hence, $\phi_0(0) = 0$.

2.2 Inexact linearization

Taylor expansion of (2) yields

$$y^{(\rho)}(t+\delta t) = \hat{y}(t) + b(x(t))\delta u + \Delta(x(t), \delta x) \tag{5}$$

for almost all $t \geq 0$ and small $\delta t > 0$, where $\delta u = u(t+\delta t) - u(t)$, $\delta x = x(t+\delta t) - x(t)$, and $\hat{y}$ is the ρ-th derivative of the output y, which is to be measured. The error term $\Delta(x, \delta x) = (\partial y^{(\rho)}/\partial x)\delta x + \cdots$ contains the state-dependent Taylor summands of first and higher order. Taking

$$u(t+\delta t) = u(t) + b(x(t))^{-1}[v(t+\delta t) - \hat{y}(t)] \tag{6}$$

for all $t \geq 0$ and δt results in the input-output dynamics

$$y^{(\rho)}(t+\delta t) \equiv v(t+\delta t) + \Delta(x(t), \delta x)$$

where the error term Δ acts as a disturbance. In practice, $\hat{y}$ is evaluated over a discrete time grid $\mathscr{T} = (t_0, t_1, \ldots)$ and the control input is a piecewise constant function, viz.

$$u(t_k + \delta t) = u_k + b(x_k)^{-1}[v(t_k + \delta t) - \hat{y}_k] \tag{7}$$

for $t_k \in \mathscr{T}$ and $\delta t \in (0, \tau]$, where $x_k = x(t_k)$, $u_k = u(t_k)$, $\hat{y}_k = \hat{y}(t_k)$, and $\tau = t_{k+1} - t_k$ is the (constant) sampling time of $\mathscr{T}$. The pseudo-input $v(\cdot)$ is now defined as piecewise constant feedback law based on the sampled external states $s_2(t_k)$. In between samples, the offset between the linearized model in (7) and the continuous-time output (2) leads to an error term Δ_k in the external co-dynamics.

Choosing $v(t) \equiv K s_2(t_k)$ for $t \in (t_k, t_{k+1}]$, the closed-loop co-dynamics are

$$\dot{s}_1(t) = \phi(s_1(t), s_2(t)) \tag{8a}$$
$$\dot{s}_2(t) = A_\text{i} s_2(t) + B_\text{i} K s_2(t_k) + B_\text{i} \Delta_k(t - t_k) \tag{8b}$$

for almost all $t \in (t_k, t_{k+1}]$ with $\Delta_k : t \mapsto \Delta(x_k, x(t_k + t) - x_k)$.

2.3 Stability concepts

We are interested in the local asymptotic stability of (1) under incremental feedback (7) and estimating the sets of initial conditions.

Definition 1. A dynamic system $\dot{x} = \psi_0(x)$ is *asymptotically stable on* Ω if and only if there exists $\beta \in \mathscr{K}\mathscr{L}$ such that any trajectory x of ψ_0 starting in $x_0 \in \Omega$ satisfies $|x(t)| \leq \beta(|x_0|, t)$ for all $t \geq 0$.

It is well known that linear dynamic systems, if asymptotically stable, are so in a global manner. Furthermore, if a linear dynamic system is asymptotically stable its states (and outputs) are bounded on bounded inputs. This property can be extended to nonlinear systems as well:

Definition 2. A dynamic system $\dot{x} = \psi(x, u)$ is *input-to-state stable* if and only if there exists $\beta \in \mathscr{K}\mathscr{L}$ and $\gamma \in \mathscr{K}$ such that any trajectory x of ψ under control u starting in x_0 satisfies $|x(t)| \leq \beta(|x_0|, t) + \gamma(||u||_\infty)$ for all $t \geq 0$.

Clearly, an input-to-state stable dynamic system is asymptotically stable on zero input. Somewhat surprising though, it can also be proven [19] that a globally asymptotically stable dynamic system is locally input-to-state stable. The function γ in the definition above is called *gain*, and it plays an important role in the stability of interconnected systems.

Theorem 1. *Let the dynamic systems* $\dot{x}_1 = \psi(x_1, v_1, u_1)$ *and* $\dot{x}_2 = \phi(x_2, v_2, u_2)$ *be input-to-state stable with gains* γ_1 *and* γ_2*, respectively; if*

$$\max\{(\gamma_1 \circ \gamma_2)(r), (\gamma_2 \circ \gamma_1)(r)\} < r \tag{9}$$

for all $r > 0$*, then the interconnection*

$$\dot{x}_1 = \psi_1(x_1, x_2, u)$$
$$\dot{x}_2 = \psi_2(x_2, x_1, u)$$

with common input u *is input-to-state stable.* $\triangleleft$

This result is known as small-gain-theorem [20]. A special case occurs if the connection of two systems is unidirectional (think either $\gamma_1 \equiv 0$ or $\gamma_2 \equiv 0$). Such a cascade of input-to-state stable systems is always input-to-state stable as well [21].

Assumption 1. The internal dynamics $\dot{s}_1 = \phi(s_1, u)$ are input-to-state stable.

Asymptotic stability of the zero dynamics is a classical precondition for exact linearization [1]. With the additional assumption of input-to-state stability, though a strong one, an appropriately chosen pseudo-input feedback gain K guarantees stability in the case of exact linearization. Since we are interested in the impact of the error term Δ onto stability in the case of inexact linearization, Assumption 1 is reasonable.

2.4 Finite-horizon output gains

We make use of two metrics in order to prove closed-loop input-to-state stability of the inexact linearization. Consider an operator $G : \mathbb{S}_1 \times S_2 \to \{[0,H] \to \mathbb{S}_2\}$ with finite horizon $H > 0$ defined by the nonlinear differential equation

$$\dot{\xi}(t) = f_G(\xi(t), s_1, s_2) \tag{10a}$$
$$e(t) = h_G(\xi(t)) \tag{10b}$$

for almost all $t \in [0,H]$ and all $(s_1,s_2) \in \mathbb{S}$. We require that $f_G : \Xi \times \mathbb{S} \to \Xi$ and $h_G : \Xi \to \mathbb{S}_2$ are Lipschitz continuous with $f_G(0,s) \equiv 0$ and $h_G(0) \equiv 0$. Thus, the finite-time response $e(\cdot)$ is well defined and finite on bounded inputs. The *finite-horizon $\mathscr{L}_p$-to-$\mathscr{L}_\infty$ gain* is

$$||G||_{p,q}^{[0,H]} = \sup_{(s_1,s_2)\in\mathbb{S}} \frac{||e(\cdot)||_q}{||s_1||_p + ||s_2||_p} \quad \text{s.t. (10) with } \xi(0) = 0 \text{ and } 0 < |(s_1,s_2)| \le r$$

for some $r > 0$ and $p, q \in \mathbb{N} \cup \{\infty\}$. By the assumptions made above, the $\mathscr{L}_p$-to-$\mathscr{L}_q$ gain is finite for any fixed $r > 0$ and $H > 0$ since (s_1,s_2) is bounded. Furthermore, the *finite-horizon $\mathscr{L}_p$-to-Euclidean gain* is

$$||G||_{p,E}^{[0,H]} = \sup_{(s_1,s_2)\in\mathbb{S}} \frac{|e(H)|}{||s_1||_p + ||s_2||_p} \quad \text{s.t. (10) with } \xi(0) = 0 \text{ and } 0 < |(s_1,s_2)| \le r$$

for some $r > 0$ and $p \in \mathbb{N} \cup \{\infty\}$. The $\mathscr{L}_p$-to-Euclidean gain is connected to reachability in nonlinear systems [18].

3 Stability Analysis

Our approach is based on approximating the gain of the inexact external dynamics (8b) subject to the error term Δ_k. To this extent, suppose an incremental feedback law has been designed with $\tau > 0$ fixed and K chosen such that all poles of the nominal closed-loop external dynamics with zero-order hold of length τ (the sampling period),

$$\bar{A}_\mathrm{i} = \left(\exp(A_\mathrm{i}\tau) + \int_0^\tau \exp(A_\mathrm{i}(\tau - t))\mathrm{d}t \, B_\mathrm{i} K \right)$$

are inside the right-half of the unit cycle.[2] After one sampling period, the external state reads

$$s_2(t_k + \tau) = \bar{A}_\mathrm{i} s_2(t_k) + \int_0^\tau \exp A_\mathrm{i}(\tau - t) \cdot B_\mathrm{i} \Delta_k(t) \mathrm{d}t \tag{11}$$

with $\Delta_k(0) = 0$ for all $k \ge 0$. The integral in (11) accounts for the error $e(\tau)$ of the inexact linearization; we want to characterize this error using finite-horizon gains. To that extent, we define an operator E on the co-states $s(t_k)$ at time t_k of which the output corresponds to $e(\tau)$. Given $\bar{s} \in \mathbb{S}$, consider E given as

$$\dot{\xi}_0(t) = A_\mathrm{i}\xi_0(t) + B_\mathrm{i}\Delta\big(T^{-1}(\bar{s}), T^{-1}(\bar{s}_1 + \xi_1(t), \bar{s}_2 + \xi_2(t) + \xi_0(t)) - T^{-1}(\bar{s})\big) \tag{12a}$$

[2] That is, any eigenvalue λ of $\bar{A}_\mathrm{i}$ satisfies $\Re\lambda > 0$ and $|\lambda| < 1$.

with output $e(t) = \xi_0(t)$, subject to the dynamics

$$\dot{\xi}_1(t) = \phi(\bar{s}_1 + \xi_1(t), \bar{s}_2 + \xi_2(t) + \xi_0(t)) \tag{12b}$$

$$\dot{\xi}_2(t) = A_\mathrm{i}(\bar{s}_2 + \xi_2(t)) + B_\mathrm{i} K \bar{s}_2 \tag{12c}$$

for almost all $t \in [0, \tau]$, where the operator states are reset to $\xi(t_k) = 0$ for any $k \geq 0$. That is, if $\bar{s} = s_k$, then $s_2(t_{k+1}) = \bar{A}_\mathrm{i} s_k + e(t_{k+1})$, where $e(t_{k+1})$ is the output of E at time t_{k+1}.

The dynamics in (11) can be viewed as sampled-data system under disturbance $e(\cdot)$. At any sampling point $k \geq 0$, we have that

$$|s_2(t_k)| = \left|\bar{A}_\mathrm{i}^k s_2(t_0) + \bar{A}_\mathrm{i}^{k-1} e(t_1) + \cdots + \bar{A}_\mathrm{i} e(t_{k-1}) + e(t_k)\right|$$
$$\leq \mu \exp(-\ell k)|s_2(t_0)| + \frac{\mu}{1 - \exp(-\ell)} \sup_{\varkappa \in (0,k]} |e(t_\varkappa)|$$

where $\mu, \ell > 0$ satisfy $||\bar{A}_i^k||_2 \leq \mu \exp(-\ell k)$ for all $k \geq 0$. Moreover, at any time $\delta t \in (0, \tau]$ in between sample points the dynamics yield

$$|s_2(t_k + \delta t)| = \left|\left(\exp(A_\mathrm{i}\delta t) + \int_0^{\delta t} \exp(A_\mathrm{i}(\delta t - t))\mathrm{d}t\, B_\mathrm{i} K\right) s_2(t_k) + e(t_k + \delta t)\right|$$
$$\leq \alpha |s_2(t_k)| + \sup_{t \in (t_k, t_k + \delta t]} |e(t)|$$

where $\alpha > 0$ satisfies $||\exp(A_\mathrm{i}\delta t) + \int_0^{\delta t} \exp(A_\mathrm{i}(\delta t - t))\mathrm{d}t\, B_\mathrm{i} K||_2 \leq \alpha$ for all $\delta t \in (0, \tau]$. Combining these inequalities, we obtain

$$|s_2(t)| \leq \alpha\mu \exp(-\ell k)|s_2(t_0)| + (1 + \frac{\alpha\mu}{1 - \exp(-\ell)})||e(\cdot)||_\infty \tag{13}$$

for all $t \in (t_k, t_{k+1}]$ and $k \geq 0$. In other words, the closed-loop external dynamics are input-to-state stable with gain $\gamma_2 = 1 + \frac{\alpha\mu}{1-\exp(-\ell)}$.

The interconnection of the internal co-dynamics, closed-loop external co-dynamics, and error dynamics is shown in Fig. 1. We now make use of the finite-horizon output gain $||E||_{2,\infty}^{[0,\tau]}$ of (12) with $0 < |\bar{s}| \leq r$ as defined in the previous section for some $r > 0$. Since f, g, and T (thus ϕ and Δ) are continuously differentiable and $\bar{s}$ is bounded, the finite-horizon gains are well defined and finite for any fixed horizon. In the next section, we propose a method to estimate the finite-horizon output gains for given nonlinear dynamics.

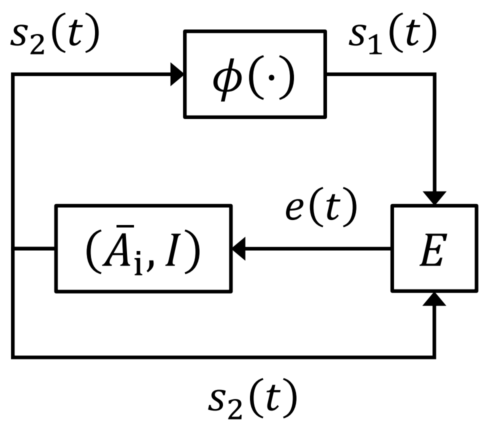

Fig. 1 Interconnection of inexact co-dynamics.

Suppose that $||E||_{2,\infty}^{[0,\tau]} \leq \gamma$ for some $r > 0$ and $\gamma > 0$. Then

$$|e(t)| \leq ||e(\cdot)||_\infty \leq \gamma||s_1(\cdot)||_\infty + \gamma||s_2(\cdot)||_\infty \tag{14}$$

if $|s(t)| \leq r$ for all $t \geq 0$.

Proposition 1. *Let $r > 0$; suppose Ω is an invariant subset of $\mathscr{B}_r(\mathbb{S})$ and $||E||_{2,\infty} \leq \gamma$ on $|s| \leq r$; if both $\gamma\gamma_2 < 1$ and $\gamma(\gamma_1\gamma_2) < 1$, where γ_1 is the ISS gain of the internal dynamics, then* (8) *is asymptotically stable on Ω.*

Proof. The closed-loop inexact linearization is composed as interconnection (Fig. 1). From (13), we have that

$$|s_2(t)| \leq \beta_2(|s_2(0)|, t) + \gamma_2||e(\cdot)||_\infty$$

with $\beta_2 \in \mathscr{K}\mathscr{L}$ for all $s_2, e : [0,\infty) \to \mathbb{S}_2$ and $t \geq 0$, which, together with (14) and the small-gain theorem implies that

$$|s_2(t)| \leq \beta'(|s_2(0)|, t) + \gamma\gamma_2||s_1(\cdot)||_\infty \tag{15}$$

with $\beta' \in \mathscr{K}\mathscr{L}$ for all $s : [0,\infty) \to \mathscr{B}_r(\mathbb{S})$ since $\gamma\gamma_2 < 1$. Eq. (15), again with the small-gain theorem, implies

$$|s(t)| \leq \beta(|s(0)|, t)$$

with $\beta \in \mathscr{K}\mathscr{L}$ for all $s : [0,\infty) \to \mathscr{B}_r(\mathbb{S})$ since $\gamma(\gamma_1\gamma_2) < 1$. As any trajectory starting in Ω remains in $\mathscr{B}_r(\mathbb{S})$, the interconnection is asymptotically stable on Ω. □

Corollary 1. *Under the assumptions of Proposition 1, the system in* (1) *under incremental feedback* (7) *is asymptotically stable on $T^{-1}(\Omega)$.* ◁

4 Solving for Output Gains

The finite-horizon gain can be evaluated by solving an associated nonlinear optimal control problem similar to [18]. Namely, consider the cost function

$$J(\bar{s}) = \kappa(\xi(H)) + \int_0^H \ell(\xi(t))\mathrm{d}t - \mu(\bar{s}) \tag{16}$$

subject to (10) with $\xi(0) = 0$. For the $\mathscr{L}_p$-to-$\mathscr{L}_\infty$ gain, choosing $\kappa \equiv 0$, $\ell(\xi) = |h_G(\xi)|^q$, and $\mu(s_1, s_2) = (\gamma||s_1||_p + \gamma||s_2||_p)^q$ we obtain

$$J(\bar{s}) \leq 0, \quad \text{i.e., } ||h_G \circ \xi||_q \leq \gamma(||\bar{s}_1||_p + ||\bar{s}_2||_p),$$

for all $|\bar{s}| \leq r$ if and only if $||G||_{p,q}^{[0,H]} \leq \gamma$. In the case that $q = \infty$, the optimal value of J then corresponds to minimizing

$$\sup_{t \in [0,H]} |\xi(t)| + \gamma(||\bar{s}_1||_p + ||\bar{s}_2||_p)$$

which is the desired norm $||\cdot||_\infty$. For the $\mathscr{L}_p$-to-Euclidean gain, choose $\kappa(\xi) = |h_G(\xi)|$, $\ell \equiv 0$, and $\mu(s_1, s_2) = \gamma(||s_1||_p + ||s_2||_p$.

The following result is a nonlinear version of [18, Theorem 1].

Theorem 2. *Let $r > 0$ and suppose that κ, ℓ, and μ be absolutely continuous; the following statements are equivalent:*

1) $J(\bar{s}) \le 0$ for all $|\bar{s}| \le r$.
2) There exists an absolutely continuous function $V : [0,H] \times \Xi \times \mathbb{S} \to \mathbb{R}$ satisfying (in the viscosity sense)

$$V(H,\xi,\bar{s}) \ge \kappa(\xi) - \mu(\bar{s}) \tag{17a}$$

$$\frac{\partial}{\partial t}V(t,\xi,\bar{s}) + \langle \frac{\partial}{\partial \xi}V(t,\xi,\bar{s}), f_G(\xi,\bar{s})\rangle + \ell(\xi) \le 0 \tag{17b}$$

$$V(0,0,\bar{s}) \le 0 \tag{17c}$$

for all $t \in [0,H]$, $\xi \in \Xi$, and $|\bar{s}| \le r$.

Theorem 2 gives necessary and sufficient conditions for a finite-horizon gain to be locally upper-bounded by $\gamma > 0$. The proof is based on textbook results for optimal control and dissipativity. It illustrates our methodology for the computation of the output gains.

Proof. Let $r > 0$.

"1) $\Rightarrow$ 2)" Define the value function

$$V^*(t_0,\xi_0,\bar{s}) = \kappa(\xi(H)) + \int_{t_0}^{H} \ell(\xi(t))\mathrm{d}t - \mu(\bar{s}) \quad \text{s.t. (10) with } \xi(t_0) = \xi_0$$

for any $t_0 \in [0,H]$, $\xi_0 \in \Xi$, and $|\bar{s}| \le r$. The assumptions on the dynamics f_G and the cost functions κ, ℓ, and μ imply absolute continuity of V^*. A classical result of optimal control theory states that V^* is the viscosity solution to the partial differential equation

$$\frac{\partial}{\partial t}V(t,\xi,\bar{s}) + \langle \frac{\partial}{\partial \xi}V(t,\xi,\bar{s}), f_G(\xi,\bar{s})\rangle + \ell(\xi) = 0 \tag{18}$$

for all $t_0 \in [0,H]$ and $\xi_0 \in \Xi$. Moreover, $V(H,\xi,\bar{s}) = \kappa(\xi) + \mu(\bar{s})$ and $V(0,0,\bar{s}) = J(\bar{s}) \le 0$, completing the result.

"2) $\Rightarrow$ 1)" Suppose V satisfies (17) for all $t \in [0,H]$, $\xi \in \Xi$, and $|\bar{s}| \le r$. Take $\xi(\cdot)$ to be the solution of (10) with $\xi(0) = 0$ for some $|\bar{s}| \le r$. Integration of (17b) along $\xi(t)$ for $t \in [0,H]$ yields

$$V(H,\xi(H),\bar{s}) - V(0,\xi(0),\bar{s}) + \int_0^H \ell(\xi(t))\mathrm{d}t \le 0$$

Using the inequalities (17a) and (17c) with $\xi(0) = 0$, we thus obtain

$$\kappa(\xi(H)) + \int_0^H \ell(\xi(t))\mathrm{d}t - \mu(\bar{s}) \le 0$$

the desired result. □

Given a desired upper bound γ, an upper bound J' on the cost function J can be found by solving for the storage function in (17), e.g., through sum-of-squares methods [22–24]. Another option is to solve the partial differential equation in (18) with techniques similar to those arising in the context of Hamilton-Jacobi-Bellman problems [25, 26]. Given $J'(\cdot) \succeq J(\cdot)$, if

$$\sup_{|\bar{s}| \leq r} J'(\bar{s}) \leq 0 \tag{19}$$

then $||G||_{p,q}^{[0,H]} \leq \gamma$ (respectively, $||G||_{p,E}^{[0,H]} \leq \gamma$) on $\mathscr{B}_r(\mathbb{S})$. Furthermore, since the cost in (16) only depends on the choice of $\bar{s} \in \mathbb{S}_1 \times S_2$, we can also sample $J(\cdot)$ over a finite sample $S \subset \mathbb{S}$. This leads to a probabilistic bound [27] for r satisfying (19).

5 Numerical Example

We present a simple example, namely, the forced unstable Van-der-Pol oscillator, as proof of concept. This example demonstrates that, despite being simple, we cannot expect an INDI feedback to be globally stable. Moreover, linearity of the internal dynamics allows to obtain the input-to-state stability gain immediately, without having to resort to numerical methods such as sum-of-squares programming [28]. Consider the dynamics given by

$$\dot{x}(t) = (x_2(t), -(1-x_1(t)^2)x_2(t) - x_1(t) + u(t)) \tag{20a}$$

$$y(t) = x_1(t) + x_2(t) \tag{20b}$$

for almost all $t \geq 0$. The first derivative of $y(\cdot)$ by time reads

$$\dot{y}(t) = \underbrace{-(1-x_1(t)^2)x_2(t) - x_1(t) + x_2(t)}_{=a(x(t))} + u(t) \tag{21}$$

for almost all $t \geq 0$ with $b \equiv 1$. Comparing (20) and (21) suggests the diffeomorphism $T : (x_1, x_2) \mapsto (x_1, x_1 + x_2)$; taking $s \equiv T(x)$ and $u(t) \equiv v(t) - a(x(t))$, we obtain

$$\dot{s}(t) = T(\dot{x}(t)) = T(x_2(t), a(x(t)) - x_2(t) + u(t)) = (x_2(t), v(t))$$

and

$$y = x_1 + x_2 = s_2$$

since $s_1 = x_1$ and $s_2 = x_1 + x_2$. By Taylor expansion, we obtain the error term

$$\Delta : (x, \delta x) \mapsto (2x_1x_2 - 1)\delta x_1 + x_1^2 \delta x_2 + x_2 \delta x_1^2 + 2x_1 \delta x_1 \delta x_2 + \delta x_1^2 \delta x_2$$

and choose $\tau = 0.1\,\mathrm{s}$ and $K = -1$.

Clearly, the internal dynamics are input-to-state stable with $\beta_1(r,t) \equiv \exp(-t)r$ and $\gamma_1 = 1$. The closed-loop external dynamics under disturbance $e(\cdot)$ read

$$s_2(t) = (1 - Kt)s_2(t_k) + e(t_k)$$

for all $t \in (t_k, t_{k+1}]$ and $k \geq 0$; hence,

$$|s_2(t_k)| \leq (1 - K\tau)^k |s_2(t_0)| + \frac{1}{K\tau} ||e||_\infty \tag{22}$$

$$|s_2(t_k + \delta t)| \leq \quad |s_2(t_k)| + \quad ||e||_\infty \tag{23}$$

for all $k \geq 0$ and $\delta t \in (0, \tau]$, implying input-to-state stability with gain $\gamma_2 = 1 + (K\tau)^{-1}$. From the sufficient conditions in Proposition 1 we have that the $\mathscr{L}_2$-to-$\mathscr{L}_\infty$ gain of E should have an upper bound

$$\gamma < (\gamma_2)^{-1} = 0.0909$$

on some $\mathscr{B}_r(\mathbb{S})$ with $r > 0$.

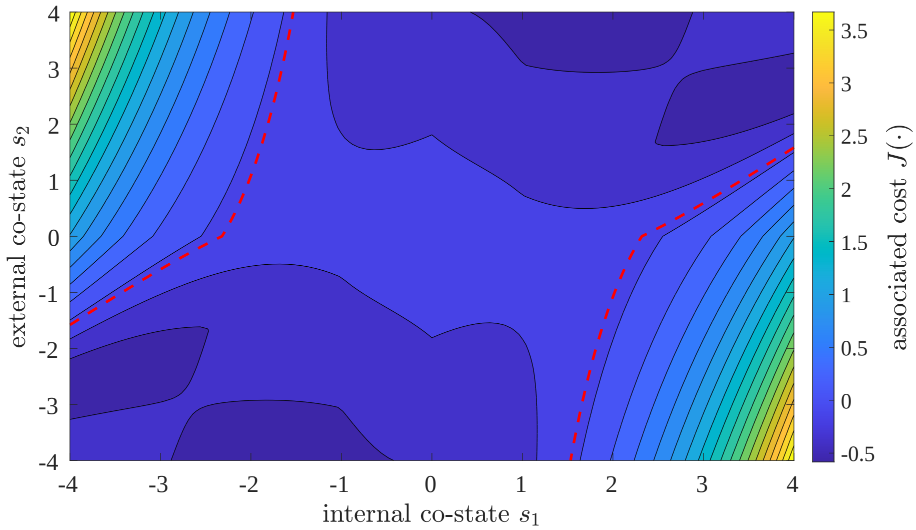

Fig. 2 Cost $J(\cdot)$ associated with $||E||_{2,\infty}^{[0,\tau]}$ with zero level set dashed in red.

To that extent, the error dynamics E are

$$\begin{aligned}\dot{\xi}_0(t) &= \Delta((\bar{s}_1, \bar{s}_2 - \bar{s}_1), (\xi_1(t), \xi_0(t) - \xi_1(t) + \xi_2(t))) \\ &= (2\bar{s}_1(\bar{s}_2 - \bar{s}_1) + (\bar{s}_2 - \bar{s}_1)\xi_1(t) - 1)\xi_1(t) + (\bar{s}_1 + \xi_1(t))^2(\xi_0(t) - \xi_1(t) + \xi_2(t))\end{aligned}$$

and

$$(\dot{\xi}_1(t), \dot{\xi}_2(t)) = (\bar{s}_2 - \bar{s}_1 + \xi_0(t) - \xi_1(t) + \xi_2(t), -\bar{s}_2)$$

recalling that $T^{-1} : (s_1, s_2) \mapsto (\bar{s}_1, \bar{s}_2 - \bar{s}_1)$. The partial differential equation (18) associated with the $\mathscr{L}_2$-to-$\mathscr{L}_\infty$ gain of E is

$$\begin{aligned}\dot{V}(t, \xi, \bar{s}) + \max\left\{0, \langle \nabla V(t, \xi, \bar{s}), \dot{\xi} \rangle\right\} = 0 \\ V(\tau, \xi, \bar{s}) = |\xi| + \gamma(||\bar{s}_1||_2 + ||\bar{s}_2||_2)\end{aligned}$$

on $(t, \xi) \in [0, \tau] \times \Xi$ (see [26] for details on viscosity solutions for the supremum norm). Since we are only interested in the cost for $\xi(0) = 0$, that is, $J(\bar{s}) \equiv V(0, 0, \bar{s})$, we can instead numerically evaluate the dynamics of E. The result for J on an equally spaced grid $[-4, 4] \times [-4, 4] \subset \mathbb{S}$ with more than 10.000 samples of $\bar{s}$ is shown in Fig. 2. We then determine the largest value $r_{\max}$ such that $J(\bar{s}) \leq 0$ on any sample with $|\bar{s}| \leq r_{\max}$, viz.

$$r_{\max} = 2.2127$$

Noting that $s_1(t_k+\tau) = s_1(t_k) + \xi_1(\tau)$ and $s_2(t_k+\tau) = s_2(t_k) + \xi_2(\tau) + e(\tau)$ for all $k \geq 0$, if $\xi(0) = 0$ and $\bar{s} = s_k$, inspection of the simulated error dynamics verifies that

$$s(t_k) \in \mathscr{B}_{r_{\max}}(\mathbb{S}) \Longrightarrow s(t_{k+1}) \in \mathscr{B}_{r_{\max}}$$

along any solution to the co-dynamics; that is, $\Omega = \mathscr{B}_{r_{\max}}$ is an invariant set in the sense of Proposition 1. According to Corollary 1, the estimated stable domain $T^{-1}(\Omega)$ of the Van-der-Pol oscillator under inexact feedback linarization is shown in Fig. 3 and compared to simulated closed-loop responses.

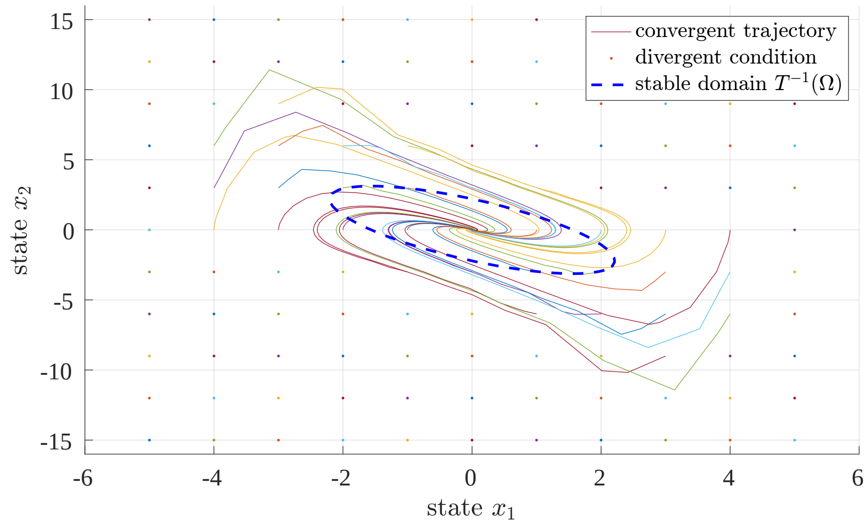

Fig. 3 Simulated system responses with estimated stable domain $T^{-1}(\Omega)$ dashed in blue.

6 Discussion

The small-gain theorem provides an upper bound on how much systems in interconnection disturb each other, thereby disregarding possibly stabilizing feedback. Hence, the domain guaranteed to be stable by the small-gain condition is by its nature a conservative estimate. On the other hand, the small-gain theorem is but one instance of a larger class of multipliers [29] for the analysis of systems subject to components that are unknown or otherwise hard to model and analyze. Here, the approach presented in this paper provides ample room for improved stability estimates in future work. Thus, by the separation, the application to nonlinear flight control would include a numerical or data-driven analysis of the internal dynamics only, which would be more complex on the full dynamics. In addition, the finite-horizon analysis of the error dynamics can be extended to account for uncertain models using ideas from [18, 29], again a hard task on the closed-loop dynamics with its combined continuous and discrete-time features. Finally, the notion of input-to-state stability allows for results on cascaded feedback [12] based on separate analysis of each loop.

Acknowledgments

The author would like to thank Werner Grimm and Ole Pfeifle for helpful discussions and comments on the draft manuscript.

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