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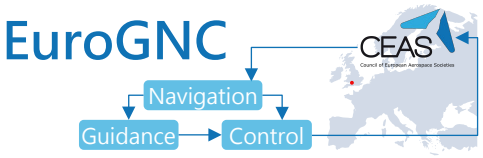

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Parametric sensitivity analysis for the robust control of uncertain space systems

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ABSTRACT

Modern modelling, analysis and robust control techniques aim to tackle parametric uncertainties as early as possible in the design of the GNC/AOCS of space systems. However, it is most often necessary to only consider a subset of the uncertain parameters to ensure the tractability of the model, and the choice is generally based on the experience of the engineer. This paper investigates the field of global sensitivity analysis to identify the most influential parameters more systematically, and more generally to improve the reliability and the efficiency of the V&V phase. The selected methods are applied to a simple satellite benchmark to capture the influence of the parameters, including their interaction effects, on the H-infinity norm. It is shown that the Morris and variance-based methods are able to rank the parameters in a complementary way, as the former is computationally cheaper while the latter quantifies the impact of each parameter on the variance of the norm. Finally, it is shown how the results of the sensitivity analysis can support the trade-off between accuracy and computation time when using mu-analysis to estimate the worst-case performance. Sensitivity analysis may also support plant identification and optimization-based methods, such as control/structure co-design, by allowing to make an informed choice on which parameters to prioritize.

Keywords: Robust control ; sensitivity analysis ; space systems

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1 Introduction

The GNC/AOCS design of space systems must take into account various uncertainties that may degrade the performance and stability. For this reason, research has been focusing on the development of modern modelling, analysis and robust control methods to tackle parametric uncertainties as early as possible in the design. These methods aim to quickly detect and correct problematic configurations before starting the more costly validation and verification (V&V) procedures, such as experimental testing or Monte-Carlo simulation campaigns, in order to avoid time-consuming reiterations during the project.

In the context of the robust control of linear or linear parameter varying (LPV) systems, the state-of-art approaches rely on the representation of the uncertain system as a Linear Fractional Transformation (LFT) model. Then, optimization techniques based on the $\mathscr{H}_\infty$ and $\mathscr{H}_2$ norms allow optimizing the controller (e.g. routine systune of the robust control toolbox of Matlab [1, 2]), and μ-analysis techniques [3] can assess the robust stability and performance over the whole parametric space. However, these techniques are limited by the number of uncertain parameters and their repetitions in the LFT model. As a consequence, it is usually necessary to only consider a subset of the uncertain parameters, in order to reduce model complexity and apply optimization-based methods. Additionally, experimental efforts may sometimes be necessary, either on ground or in flight, to perform identification of some of the parameters. In the current practice, the "most important" parameters to be included in the uncertain model or to be identified are chosen based on the experience of the engineer. This paper investigates more systematic methods to rank the parameters by order of "importance" in the context of the robust analysis and control of linear or LPV systems. The objectives are threefold: 1) improve the reliability of the V&V phase: reduce the risk of erroneously judging a parameter as non-influential, provide quantitative information to justify why each parameter is neglected or not; 2) improve the efficiency of the V&V phase: reduce the size of the parametric space to enable optimization techniques or reduce their computation time, focus the experimental identification efforts directly into the meaningful parameters; and 3) more generally, provide new insight into the robustness of the model that complements other existing approaches and validates or corrects the engineering intuition.

Sensitivity analysis (SA) is the mathematical field that studies *how uncertainty in the output of a model can be apportioned to different sources of uncertainty in the model input* [4]. Sensitivity analysis is used in various fields of natural or engineering sciences, where the resulting knowledge of the system may help establishing priorities (focus the experimental testing, the modelling or research effort, etc., on the most influential parameters), simplifying models that are computationally expensive, identifying which parameters interact with others [5]. In the context of robust control, it is worth mentioning that the mu-sensitivity [6] is sometimes used as a SA method for this purpose. However, in the context of SA (the use of mu-sensitivity to guide the optimization in mu-analysis algorithms, as in [7], is not addressed here), mu-sensitivity pertains to the class of "one-factor-at-a-time" (OAT) techniques, which consist in varying one factor at a time while keeping the others fixed. For example, the routine *wcgain* of Matlab's *robust control toolbox* computes the sensitivity as an approximation of the derivative of the worst-case gain with regard to each parameter. More generally, most OAT methods are based on the computation of the derivative of the output with regard to each uncertain parameter. SA specialists strongly advise against OAT methods, since they are not only very local (they are generally computed at either the nominal or the worst-case parametric configuration), but they also fail to capture the interactions between the parameters. In satellites, a typical interaction between parameters is the resonance that arises for some specific angular configurations of the solar array, depending on its uncertain natural frequencies and modal participation factors as well as the moment of inertia of the central body.

On the contrary, so-called global SA (GSA) methods are designed to explore the entirety of the parametric space and provide concise information about the overall influence of the parameters as well as their interactions. GSA techniques can be divided in two categories: screening techniques, which aim to identify non influential parameters with a small number of model evaluations, and techniques that aim to

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provide quantitative information based on some importance measures [8]. Screening techniques include for example Morris method (or elementary-effect method) [9], supersaturated design [10], or sequential bifurcation [11]. Sampling-based methods relying on the fitting of a linear model can use measures such as Pearson correlation coefficient, standard regression coefficient, or standardized rank regression coefficient among others [8], as long as the influence of the parameters on the output is monotonous. Methods based on the decomposition of the variance aim to quantify the impact of the parameters on the variance of the output; such indices can be computed with Monte-Carlo sampling [12], but improvements regarding computation time have been proposed using quasi-random sequences, bootstrap methods, or asymptotic approximations of the variance [13]. Finally, so-called moment-independent measures can be used when the variance is not a good representation of the variability of the distribution [14]. In this paper, three usual GSA techniques were selected following [4, 5], namely linear regression, Morris method, and variance-based analysis; they will be investigated in the context of the robust control synthesis and analysis. It is worth mentioning that a similar approach using the Morris method was already mentioned in [15, 16].

This paper is organized as follows. Section 2 introduces a satellite benchmark. Section 3 presents a review of the global SA techniques and their application to the benchmark. Finally, Section 4 discusses how the results of the SA can be used in a search of the worst-case performance with μ-analysis.

2 Presentation of the use-case: a satellite benchmark

The use-case is a satellite composed of a central rigid body, and a flexible solar panel. The dynamic model is obtained under LFT representation with the Satellite Dynamics Toolbox - Library (SDTlib) [17]. A controller was synthesized similarly to [18]. There are 20 uncertain parameters, detailed in Table 1, all with uniform distribution of probability. The uncertainties on the central body are chosen large, as in early phases of a project when the mechanical design is not yet fixed, or to take into account variations of the propellant mass during the mission. The delay is modeled with a 2-nd order Padé approximation, written as an LFT model.

Remark: the solar array angle θ is not an uncertain parameter, but it is considered as such in the LFT form for classical robust control tools, and in this study to apply the global sensitivity analysis techniques. Its range of variation is considered between $-\pi/2$ and $+\pi/2$ for symmetry reasons. The LFT representation is parameterized with $t = \tan(\theta/2)$ [19].

Element	Parameters	Uncertainty
Central body	Center of mass (x,y,z)	$\pm$ 5 or 10 cm
	Moments of inertia (x,y,z,xy,xz,yz)	$\pm$ 20%
	Mass	$\pm$ 20%
Solar array	Natural frequencies (3 modes)	$\pm$ 20 %
	Modal participation factors (5 in total)	$\pm$ 20 %
	Angle θ	$\in [-\pi/2, +\pi/2]$
Avionics	Command delay	$\in [1, 30]$ ms

Table 1 Uncertain parameters

In this paper, we are interested in the transfer function represented in Figure 1, representing the closed-loop sensitivity function, whose $\mathscr{H}_\infty$ norm is the inverse of the modulus margin of the system.

Depending on the parametric configuration, the $\mathscr{H}_\infty$ norm can be determined either by the attitude control mode around 1 rad/s (in 98% of the configurations), or by the first flexible mode around 6 to 10 rad/s (in 2% of the configurations). This is also visible on Figures 2a and 2b that present the

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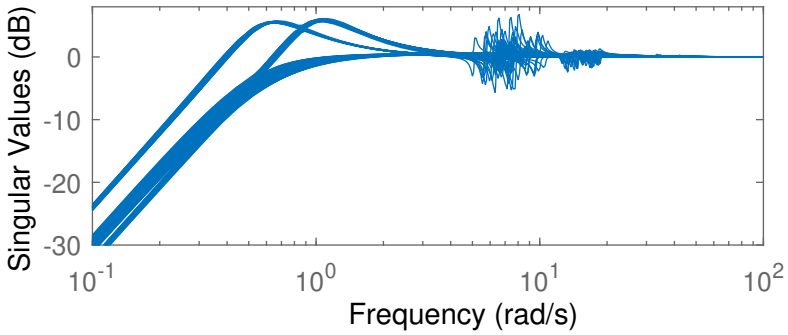
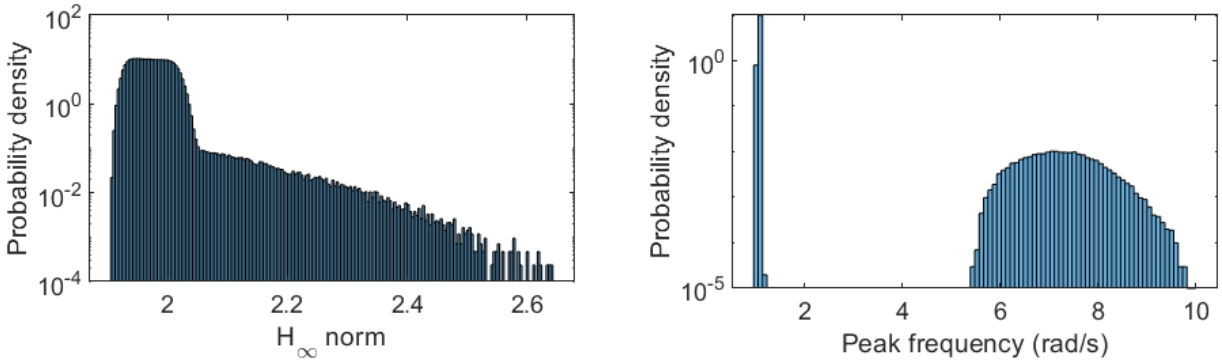


Fig. 1 Singular values of the 3×3 benchmark transfer function (50 samples)

probability distribution of the $\mathscr{H}_\infty$ norm and of the corresponding peak frequency, respectively (obtained by evaluating $\sim 10^6$ random samples and plotting the corresponding histograms). It is sought to study the sensitivity of the $\mathscr{H}_\infty$ norm to the uncertain parameters and in particular to take into account the presence of these two candidates worst-case peaks.



(a) Distribution of the $\mathscr{H}_\infty$ norm **(b) Distribution of the peak frequency**

Fig. 2 Distributions of probability relative to the $\mathscr{H}_\infty$ norm of the benchmark (log scales)

3 Global sensitivity analysis: review and application

3.1 Notations

Let us consider the general notations:

- X_i the random variable representing the i-th uncertain parameter, $1 \leq i \leq k$, with any given distribution of probability,
- $\mathbf{X}$ the vector containing the k variables X_i, also called the input parameters,
- $Y = f(\mathbf{X})$ the random variable representing the output of the evaluation of the model, here Y is the $\mathscr{H}_\infty$ norm of the uncertain system.

Note that a "model evaluation" consists in (i) randomly selecting a parametric configuration, (ii) substituting this configuration in the LFT model (Matlab's function *usubs*), and (iii) evaluating the $\mathscr{H}_\infty$ norm. The reviews presented in this section are largely based on the references [4, 5].

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3.2 Linear regression

3.2.1 Linear regression: review

The linear regression method constructs a linear relationship between Y and the parameters X_i, and uses the coefficients b_i as measures of the sensitivities $S_{\text{lr}}(X_i)$:

$$\hat{Y} = b_0 + \Sigma b_i X_i \quad , \qquad S_{\text{lr}}(X_i) = \left| b_i \frac{s_i}{s} \right| \tag{1}$$

where s_i and s are the estimated standard deviation of X_i and Y respectively. The b_i are evaluated with the least-squares method from a set of N Monte-Carlo samples of the parametric configurations.

The linear regression method is intuitive and easy to implement. It also provides the sign of the effect of an input parameter on the output. Despite being simple, it explores the entire domain of definition of each parameter. However, the linearity assumption is not always adequate. The method provides an index R^2 indicating the degree of linearity of the system:

$$R^2 = \Sigma \left(S_{\text{lr}}(X_i) \right)^2 \tag{2}$$

In [4], it is indicated to not use the linear regression to order the parameters when $R^2 < 0.3$, and that the linear model can be considered as good when $R^2 > 0.7$ (but this point will be discussed in the application). Note that the R^2 accounts for individual nonlinearities of the parameters and for interaction between parameters without being able to distinguish. Finally, it can be mentioned that the sensitivity indices provided by the linear regression can be used to rank the parameters, but they provide very little quantitative information about the impact of the parameters on the output.

3.2.2 Linear regression: application

Method

The use-case model was evaluated over a set of $N = 10000$ randomly sampled parametric configurations (computation time: 161s on a 12th Gen Intel(R) Core(TM) i7-12700H 2.30 GHz), and the coefficients of the linear model were computed by a least-squares method. All uncertain parameters were normalized such that their values are between -1 and $+1$.

Results

Figure 3a shows the resulting scatter plot for the parameter "dt" (command delay), which has the highest sensitivity index. While the linear regression accurately captures a strong correlation between higher values of "dt" and higher values of the $\mathscr{H}_\infty$ norm, the linear assumption is not sufficient to explain that the dispersion also seems to increase with "dt". This is even more visible on Figure 3b with the parameter "w1" (frequency of the first flexible mode of the solar array), ranked 4th by decreasing sensitivity. Moreover, the linear regression is not adequate to capture the non-monotonous influence of the angle of the solar array (parameter "theta"), as visible in Fig.4; as a result, this parameter is ranked 18th in decreasing sensitivity, which is not satisfying.

The R^2 is 0.71, which is generally considered as good in sensitivity analysis. This value is essentially obtained because the linear assumption is indeed good in the 98% of the cases where the $\mathscr{H}_\infty$ norm is determined by the attitude control mode around 1 rad/s. However, the method is inadequate to capture the presence of several candidate worst-case peaks, even when they are not very rare (the flexible mode determines the $\mathscr{H}_\infty$ norm in 2% of the cases and can be critical for space systems). Nonetheless, the method provides a quick visual idea of the influence of the parameters whose influence is, if not linear, at least monotonous.

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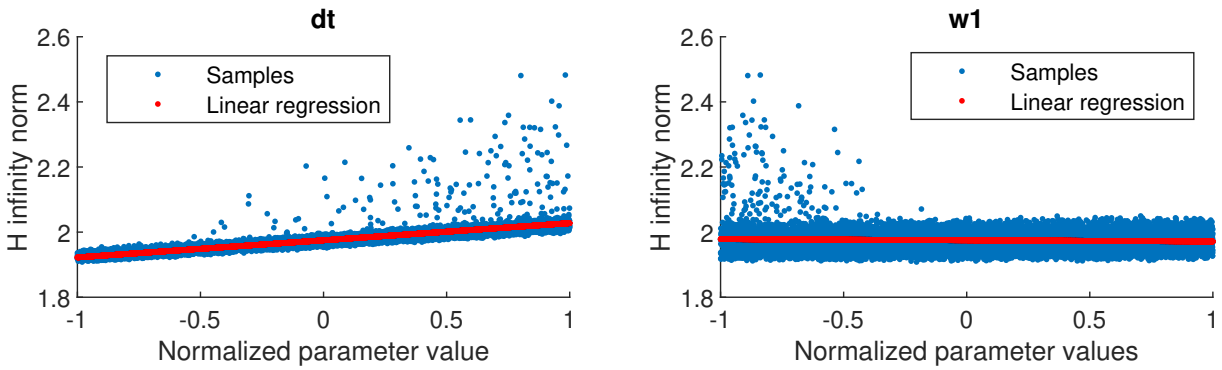

(a) Command delay "dt" **(b) Frequency of first flexible mode "w1"**

Fig. 3 Scatter plot and linear regression for 2 of the 20 parameters

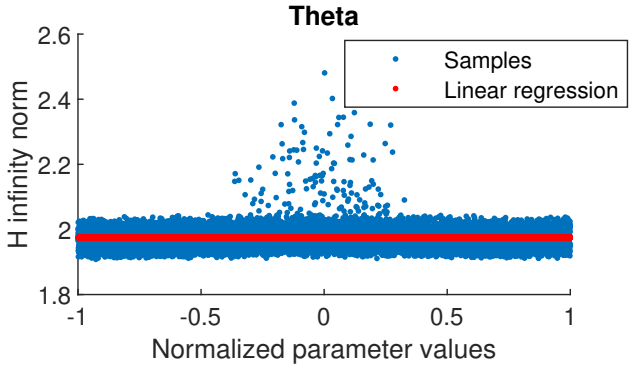

Fig. 4 Scatter plot and linear regression for the solar array angle "theta"

3.3 Morris method

3.3.1 Morris method: review

The Morris method, or elementary effects method, is general, cheap in computations, and is easy to implement. Its only drawback is that the information that it produces is not as quantified as with the variance-based method presented in Section 3.4.

Elementary effects

The Morris method relies on the computation of "elementary effects". Elementary effects, noted $d_i(\mathbf{X})$, measure the effect on the output, at a given point $\mathbf{X}$ of the parametric space, of varying the i-th parameter by a value Δ:

$$d_i(\mathbf{X}) = \frac{Y(X_1,...,X_{i-1},X_i+\Delta,X_{i+1},...,X_k) - Y(\mathbf{X})}{\Delta} \tag{3}$$

Each elementary is a "one-factor-at-a-time" (OAT) effect, but the sensitivity indices defined thereafter allow to explore the whole parametric space.

Sensitivity indices

For each parameter, r elementary effects will be computed at different points $\mathbf{X}_1$, ..., $\mathbf{X}_r$ of the parametric space. Two sensitivity indices are defined for each parameter, respectively the mean (of the absolute value) and the standard deviation of these r elementary effects:

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$$\mu_i^* = \frac{1}{r}\sum_{j=1}^{r}\left|d_i(\mathbf{X}_j)\right| \tag{4}$$

$$\sigma_i = \sqrt{\frac{1}{r}\sum_{j=1}^{r} d_i(\mathbf{X}_j)^2 - \left(\frac{1}{r}\sum_{j=1}^{r} d_i(\mathbf{X}_j)\right)^2} \tag{5}$$

Thus, each parameter is attributed a point in the plane (μ^*, σ). The mean μ^* is the main indicator of the sensitivity which reflects the total effect that the parameter has, in average, on the output. The standard deviation σ is a good indicator of either the nonlinear influence of the parameter or the presence of interactions with other parameters (but the method does not allow the distinction). Note that this indication is produced individually for each parameter, contrary to the linear regression which only provided an overall assessment of the linearity of the whole model with the R^2.

Sampling strategy

Since the computation of one elementary effect requires to evaluate the model in two points, a naïve strategy would require $2r$ points per parameter, hence a total of $2rk$ points for a total of rk elementary effects. The strategy developed in [9] only requires $r(k+1)$ points.

Assuming that the value of each parameter is normalized between 0 and 1, the range of variation is discretized in p levels: each parameter can only take a value in the set $\Omega = \{0, 1/(p-1), 2/(p-1), ..., 1\}$. The parametric space is thus Ω^k. It is common practice to choose p even and $\Delta = p/(2(p-1))$.

Remark: it assumes that the distribution of each parameter is uniform. If it is not the case, it is recommended in [4, 5] to choose the levels corresponding to quantiles of the distribution.

The sampling strategy consists in generating r random trajectories in the parametric space. Each trajectory is composed of $k+1$ points noted $\mathbf{X}^{(1)}, \mathbf{X}^{(2)}, \ldots\ \mathbf{X}^{(k+1)}$ such that one and only one parameter is modified by the value Δ between two successive points, and each parameter is modified exactly once on a trajectory. Thus, each trajectory allows the computation of k elementary effects (one per parameter). In practice, a trajectory is represented by an orientation matrix $\mathbf{B}^* \in \Omega^{(k+1)\times k}$, where the $k+1$ lines represent the parametric configurations $\mathbf{X}^{(1)}, \mathbf{X}^{(2)}, \ldots\ \mathbf{X}^{(k+1)}$, and each column represents a parameter. The orientation matrix can be generated with the following definition:

$$\mathbf{B}^* = \left(\mathbf{J}_{k+1,1}\mathbf{X}^* + \Delta/2\left(\left(2\mathbf{B} - \mathbf{J}_{k+1,k}\right)\mathbf{D}^* + \mathbf{J}_{k+1,k}\right)\right)\mathbf{P}^* \tag{6}$$

Where:

- $\mathbf{J}_{a,b}$ is a matrix of dimension $a \times b$ filled with ones,
- $\mathbf{X}^*$ is randomly chosen in Ω^k,
- $\mathbf{B}$ is a strictly lower triangular matrix of dimensions $(k+1) \times k$, whose non-zero elements are ones,
- $\mathbf{D}^*$ is a diagonal matrix of size k whose elements are either -1 or $+1$ (with the same probability),
- $\mathbf{P}^*$ is a random permutation matrix (each column and line contains exactly one elements equal to 1, all others are zero).

This procedure ensures a random exploration of the parametric space [9].

3.3.2 Morris method: application

Method

The values $r = 2000$ and $p = 800$ were chosen. The computation time for the $r(k+1) = 42000$ model evaluations was 927s.

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Results

The results of the analysis of the use-case are presented in Figure 5. Firstly, we notice that the abscissa of the parameter "dt" (command delay) is around 0.1: this indicates that, when the (normalized) value of "dt" is changed of $\Delta \approx 0.5$, the $\mathscr{H}_\infty$ norm was changed of $0.1 \times \Delta \approx 0.05$ on average. Moreover, the associated standard deviation is 0.05, which reflects either nonlinearity or interactions with the other parameters; however, since the value of the mean is larger than the standard deviation, it seems that the influence of "dt" is essentially additive. Most of the other parameters are distributed along the same diagonal as "dt". However, three parameters are located on the top left corner, namely the angle of the solar array ("theta"), and the modal participation factor in rotation ("Ry1") and frequency ("w1") of the first flexible mode of the solar array. Those parameters exhibit a large standard deviation of their elementary effects with regard to the mean values; this is an indicator of their coupled interaction leading to the resonance that determines the $\mathscr{H}_\infty$ norm for 2% of the parametric space.

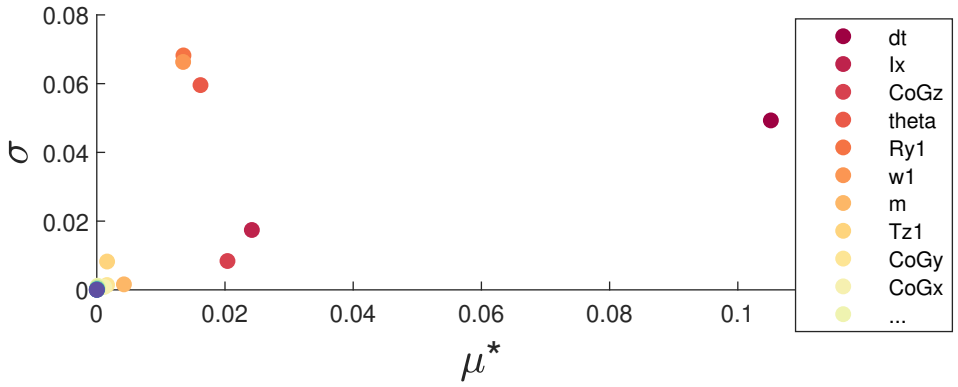

Fig. 5 Results of Morris method. The legend is ordered by decreasing μ^*.

Considered together, the two indices μ^* and σ provide information about the influence of all parameters of the system including their interactions for a cheap computational cost. However, the information it provides is essentially qualitative.

Convergence and computational cost

To study the convergence, we define the absolute error (AE) in a similar manner to [13]:

$$\mathrm{AE}(r) = \sum_{i=1}^{k} \left| \widehat{S}(i,r) - S(i,r_{\max}) \right| \tag{7}$$

where $\widehat{S}(i,r)$ is the estimated sensitivity index (either μ_i^* or σ_i) obtained using only the first r trajectories (the total number of model evaluations is $r(k+1)$, here with $k = 20$), and the index $S(i,r_{\max})$ obtained with the highest value $r_{\max} = 2000$ is used a a reference for the "true" value of the sensitivity index.

The resulting plot of AE versus r is presented in Figure 6 and suggests that the method needs at least $r = 500$ to 1000 to converge, which is also consistent with the results of [20].

3.4 Variance-based method

The variance-based method produces the most extensive information about the system. Indeed, it quantifies the contribution of each parameter to the variance of the output. As a consequence, it is also the method with the highest computation cost. For this reason, it is suggested in [21] that Morris and variance-based methods can be used in combination: Morris method provides a first screening to eliminate the least influential parameters, and then the variance-based method is performed on the reduced model to refine the analysis on the remaining parameters.

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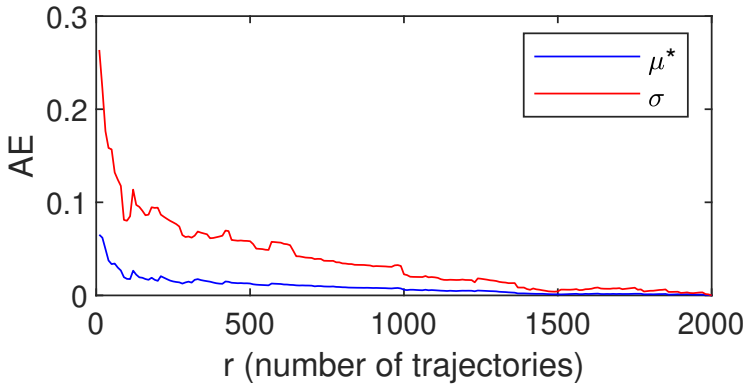

Fig. 6 Convergence of Morris method: AE vs r.

3.4.1 Variance-based method: review

Let us consider the additional notations:

- $\mathbf{X}_{\sim i}$ the vector containing the random variables except X_i,
- $V(\cdot)$ and $E(\cdot)$ the variance and expected value respectively.

Sensitivity indices

The method relies on the computation of two sensitivity indices, noted S_i and S_{Ti}. S_i is the first-order sensitivity coefficient, defined as:

$$S_i = \frac{V_{X_i}(E_{\mathbf{X}_{\sim i}}(Y|X_i))}{V(Y)} \tag{8}$$

Where:

- $E_{\mathbf{X}_{\sim i}}(Y|X_i)$ is the expected value of Y if X_i is fixed and all other parameters vary,
- The variance $V_{X_i}(\cdot)$ of the above value is computed when X_i is varying,
- $V(Y)$ is the unconditional variance of Y.

There are two equivalent interpretations of S_i:

- S_i is the expected (normalized) reduction of the variance if X_i could be fixed to its true value. This is due to the identity $V_{X_i}(E_{\mathbf{X}_{\sim i}}(Y|X_i)) + E_{X_i}(V_{\mathbf{X}_{\sim i}}(Y|X_i)) = V(Y)$.
- $V_{X_i}(E_{\mathbf{X}_{\sim i}}(Y|X_i))$ is the first-order (additive) contribution of parameter X_i to the variance of Y. This point will be more detailed in the decomposition of the variance further below.

S_{Ti} is the total-effect index, defined as:

$$S_{Ti} = \frac{E_{\mathbf{X}_{\sim i}}(V_{\mathbf{X}_i}(Y|\mathbf{X}_{\sim i}))}{V(Y)} = 1 - \frac{V_{\mathbf{X}_{\sim i}}(E_{\mathbf{X}_i}(Y|\mathbf{X}_{\sim i}))}{V(Y)} \tag{9}$$

Which can be interpreted equivalently as:

- $E_{\mathbf{X}_{\sim i}}(V_{\mathbf{X}_i}(Y|\mathbf{X}_{\sim i}))$ is the expected variance that would remain if all parameters, except X_i, could be fixed to their (unknown) true value.
- As for S_i, $V_{\mathbf{X}_{\sim i}}(E_{\mathbf{X}_i}(Y|\mathbf{X}_{\sim i}))$ is the first-order contribution of $\mathbf{X}_{\sim i}$, and thus, S_{Ti} is the contribution of all the rest, that is, all contributions of any order involving X_i. This point will be more detailed in the decomposition of the variance further below.

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Decomposition of the variance

Assuming decorrelation of the uncertain parameters, the variance of Y can be decomposed as:

$$V(Y) = \sum_i V_i + \sum_i \sum_{j>i} V_{ij} + ... + V_{12...k} \tag{10}$$

With:

- $V_i = V_{X_i}(E_{\mathbf{X}_{\sim i}}(Y|X_i))$ the first-order contributions of each parameter X_i,
- $V_{ij} = V_{X_iX_j}(E_{\mathbf{X}_{\sim ij}}(Y|X_i,X_j)) - V_{X_i}(E_{\mathbf{X}_{\sim i}}(Y|X_i)) - V_{X_j}(E_{\mathbf{X}_{\sim j}}(Y|X_j))$ the joint effects of all pairs of two parameters X_i and X_j,
- Etc for the joint effects of combinations of 3 to k parameters.

Or, after dividing by $V(Y)$:

$$\sum_i S_i + \sum_i \sum_{j>i} S_{ij} + ... + S_{12...k} = 1 \tag{11}$$

Where:

- The S_i represent indeed the first-order contribution of each X_i on the variance,
- The $S_{ij...k}$, defined by normalizing the variances $V_{ij...k}$ by $V(Y)$, represent all the interaction effects between parameters.

The S_{Ti} are the sum of all the terms involving X_i, e.g. $S_{T1} = S_1 + S_{12} + S_{13} + ... + S_{123} + S_{124} + ... + S_{12...k}$. Thus, the difference between S_i and S_{Ti} is an indication of the interaction of X_i with the other parameters.

Computation of the sensitivity indices

Noting N the base sample size that is required to evaluate a variance or an expected value, a simple strategy, using two loops to evaluate successively the variance and the expected value or vice-versa, would imply N^2 model evaluations to compute only one S_i or S_{Ti}. However, the method presented in [13] only requires $N(k+2)$ model evaluations to compute all S_i and S_{Ti} when the uncertain parameters are decorrelated.

Two matrices $\mathbf{A}$ and $\mathbf{B}$ of size $N \times k$ are generated. Each line represents a parametric configuration. The generation can be random (Monte-Carlo), or quasi-random (e.g. Sobol sequence, Halton sequence, Latin hypercube). In the latter case, the two matrices $\mathbf{A}$ and $\mathbf{B}$ can be respectively the left and right parts of a quasi-random sequence of size $N \times 2k$. It is important to note that, in this case, $\mathbf{A}$ and $\mathbf{B}$ do not play a symmetric role, because the uniformity of the quasi-random sequence decreases when the index of the column increases.

Remark: quasi-random sequences only exist for uniform distributions of probability. It may or may not be possible to transform the quasi-random sequence from uniform distribution to a given distribution of probability, depending on the sequence and on the distribution; in any case, Monte-Carlo sampling is always possible (but generally demands more samples [13]).

For each parameter X_i, a matrix $\mathbf{A}_B^{(i)}$ is generated as the matrix $\mathbf{A}$ where the i-th column is replaced by the i-th column of $\mathbf{B}$. This step is sometimes called re-sampling, and the procedure of replacing one column can be called radial sampling.

Finally, the indices of sensitivity can be computed using the expressions from [22]:

$$V_{X_i}(E_{\mathbf{X}_{\sim i}}(Y|X_i)) = V(Y) - \frac{1}{2N} \sum_{j=1}^{N} \left(f(\mathbf{B})_j - f(\mathbf{A_B}^{(i)})_j \right)^2 \tag{12}$$

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$$E_{\mathbf{X}_{\sim i}}(V_{\mathbf{X}_i}(Y|\mathbf{X}_{\sim i})) = \frac{1}{2N}\sum_{j=1}^{N}\left(f(\mathbf{A})_j - f(\mathbf{A_B}^{(i)})_j\right)^2 \quad (13)$$

Where, for example, $f(\mathbf{B})_j$ is the output of the model evaluated for the line j of matrix $\mathbf{B}$. These expressions are based on the re-writing of the integrals expressing the variance and expected values (cf. [22, 23] for demonstrations), and other expressions are also possible (cf. [13] for a review).

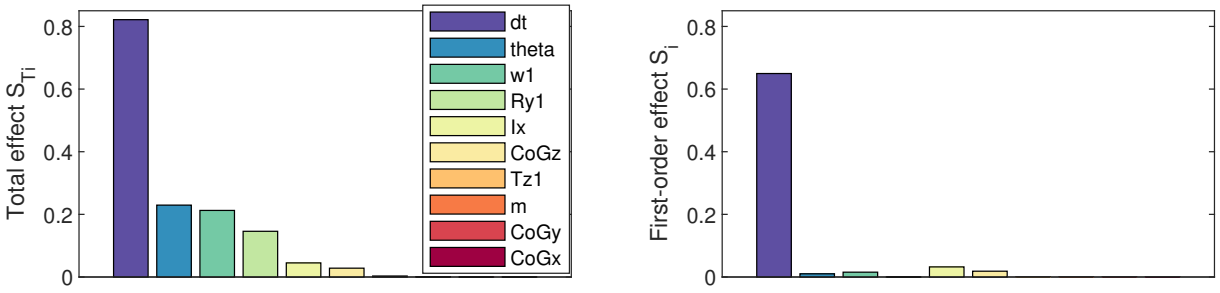
3.4.2 Variance-based method: application

Method

As suggested in [21], to limit the computation time, the variance-based method is applied to the use-case after removing the parameters identified as the least influential by the Morris method. The 10 parameters with highest μ^* and σ were kept in the model, the 10 others were set to their nominal values. The matrices $\mathbf{A}$ and $\mathbf{B}$ were generated with a quasi-random Sobol sequence with $N = 20000$. The estimators of equations (12) and (13) were used. The computation time for the $N(k+2) = 240000$ model evaluations was 3721s.

Results

The resulting total (S_{Ti}) and first-order (S_i) indices are presented in Figures 7a and 7b respectively.



(a) Total order effects (S_{Ti}). **(b) First-order effects (S_i).**

Fig. 7 Result of the variance method. The parameters are ordered by decreasing S_{Ti} in both graphs.

These plots can be interpreted as follows:

a) The value of S_i for the parameter "dt" (commande delay) is 0.65. It means that 65% of the variance of the $\mathscr{H}_\infty$ norm is due to the first-order effect of "dt". Moreover, if this parameter could be fixed, the variance of the $\mathscr{H}_\infty$ norm would be reduced of 65% (on average across all possible true values of "dt"). The same interpretation can be made for the other parameters.

b) The value of S_{Ti} for the parameter "dt" is 0.82. It means that the sum of all the contributions, in the decomposition of the variance, that involve "dt", amounts for 82% of the variance of the $\mathscr{H}_\infty$ norm. Moreover, if all parameters except "dt" could be fixed to their (unknown) true values, it would remain 82% of the variance on average. The same interpretation can be made for the other parameters.

c) For a parameter X_i, the difference between S_{Ti} and S_i provides information about the interaction of X_i with other parameters. For example, the influence of "dt" (command delay) is essentially additive (i.e. due to the first-order effect) because $S_i \approx S_{Ti}$. On the contrary, for "theta" (solar array angle), "w1" and "Ry1" (frequency and participation factor of flexible mode 1), we have $S_i << S_{Ti}$ so the total effect is essentially caused by the higher order (interaction) terms; this is consistent with the physical interpretation (the resonance occurs only for some values of these parameters) and with Morris method.

d) The indices S_i and S_{Ti} do not provide the same ranking order. The parameters "Ix" (moment of inertia around x) and "CoGz" (position of center of mass on z) have a larger first-order contribution, but

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smaller total effect than "theta", "w1" and "Ry1". The relevance of ordering by S_i or S_{Ti} depends on the application. Remark: according to the literature [4, 5], μ^* are generally a good proxy for S_{Ti}, but in this benchmark, the ranking of μ^* actually matches with S_i instead.

e) The sum of all S_i is 0.72, indicating that 72% of the variance is caused by the first-order (additive) effects (and essentially the first-order effect of "dt").

Convergence and computational cost

The absolute error (AE) is defined similarly to the Morris method:

$$\mathrm{AE}(N) = \sum_{i=1}^{k} \left| \widehat{S}(i,N) - S(i,N_{\max}) \right| \tag{14}$$

where $\widehat{S}(i,N)$ is the estimated sensitivity index (either S_{Ti} or S_i) obtained using only the first N base samples (the total number of model evaluations is $N(k+2)$, here with $k=10$), and the index $S(i,N_{\max})$ obtained with the highest value $N_{\max}=20000$ is used a a reference for the "true" value of the sensitivity.

The resulting plot of AE versus N is presented in Figure 8 and suggests that the method needs at least $N=3000$ to 5000 to converge, which is also consistent with the results of [20]. Note that even with $N=20000$, in the results presented above, some of the least influential parameters have a S_i that is slightly negative or superior to the S_{Ti}, which should not be physically possible but is due to insufficient convergence (this has no strong impact on the analysis, it should simply be considered that these S_i are very close to zero).

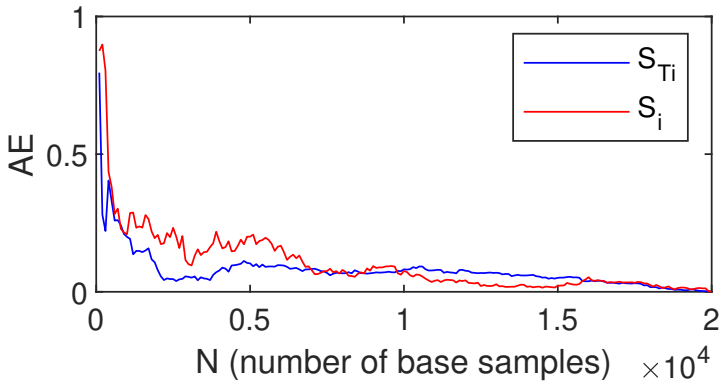

Fig. 8 Convergence of variance method: AE vs N.

"Factors prioritization" and "factors fixing" settings

Using S_{Ti} or S_i to rank the parameters depends on the reasons that motivated the sensitivity analysis.

Reference [5] describes the "factor prioritization setting" as: *identify which factor, once 'discovered' and fixed at its true value, would reduce $V(Y)$ the most.* In this case, the S_i are the relevant metric. An example of application for space systems is the plant identification, either on-ground or in-orbit. The S_i allow determining which factor(s) should be identified in priority and quantify the expected reduction of variance.

Remark: in the application presented here, the angle "theta" of the solar array was represented as an uncertain parameter with uniform distribution, although in reality, it is a time-varying parameter that will necessarily take all possible values during a mission; this point should be kept in mind, as it affects the variances that are computed by the method. If the method is now re-applied by fixing the angle "theta" to its worst-case configuration found thereafter in Section 4, the parameters "w1" and "Ry1" become 2nd and 3rd by decreasing Si (instead of "Ix" and "CoGz").

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Reference [4] defines the "Factor fixing setting" as: *identify the factor or the subset of input factors that we can fix at any given value over their range of uncertainty without significantly reducing the output variance*. In this case, the S_{Ti} should be used. This setting is relevant for model reduction to improve optimization tools. For example, the control/structure co-design approach presented in [24] used a reduced set of the uncertain and design parameters to enable the co-optimization of the controller and structural parameters; in such applications, the sensitivity analysis would allow to quickly make an informed choice about which parameters to optimize. Another optimization application is μ-analysis, as detailed in Section 4.

4 Worst-case search with sensitivity- and mu-analysis

In this section, it is proposed to use the results of the sensitivity analysis to reduce the size of the LFT model and enable the μ-analysis tools. The μ-analysis is also compared to a Monte-Carlo experiment.

The Monte-Carlo experiment simulates $N_{MC} = 1.15 \times 10^6$ random samples. N_{MC} verifies:

$$N_{MC} \geq \frac{\ln(\delta)}{\ln(1-\varepsilon)} \tag{15}$$

with the values $\delta = \varepsilon = 10^{-5}$, which guarantees the probability [25]:

$$\mathrm{P}(\mathrm{P}(Y > \hat{Y}_{\mathrm{max,MC}}) < \varepsilon) \geq 1 - \delta \tag{16}$$

that is, that the worst-case $\hat{Y}_{\mathrm{max,MC}}$ found during the Monte-Carlo experiment covers 99.999% of the probability distribution of Y with a confidence level of 99.999%. This result does not depend on the number of uncertain parameters (here we keep the initial set of 20 in the use-case).

The μ-analysis, on the other hand, is computed with a reduced number of uncertain parameters. Since this is a "factor fixing setting", the parameters are ranked by decreasing S_{Ti}, and we define a set of 9 reduced systems by keeping the first 2 to 10 parameters while the others are set to their nominal value. The μ upper bounds of these reduced systems are computed independently from each other; each parameter is initialized to its nominal value. Then, the μ lower bounds are computed with an iterative procedure: the worst-case parametric configuration found for the i-th reduced system is used to initialize the algorithm applied to the (i+1)-th reduced system (the last parameter is initialized with its nominal value); this procedure was found necessary to avoid overly optimistic lower bounds. The routine *wcgain* of Matlab's *robust control toolbox* was used (note that more recent algorithms such as [26, 27] are probably more efficient in terms of conservatism and/or computation time).

Figure 9a shows the lower and upper bounds depending on the number of parameters, and the worst-case performance found by the Monte-Carlo runs. Figure 9b shows the associated computation times. We observe the following points:

a) μ upper bounds (Figure 9a, red plot): they seem to converge to the value 2.90. With only the first 6 parameters (each total effect is at least 3% of the unconditional variance), the upper bound is already 2.84. The parameters ranked 7th to 10th (each total effect is no more than 0.3% of the unconditional variance), only slightly affect the upper bound. These upper bounds are not very conservative, since they are only slightly larger than the worst-case found by Monte-Carlo and by the μ lower bounds.

b) μ lower bounds (Figure 9a, blue plot): the evolution of the lower bound is not monotonous, which indicates that adding parameters with low influence can make the search less reliable, even with a proper initialization with the iterative procedure (in particular, for 8 parameters, the result is overly optimistic). Nonetheless, with only 7 parameters, a rare worst case that is larger than the one found by Monte-Carlo (with all 20 parameters), was identified.

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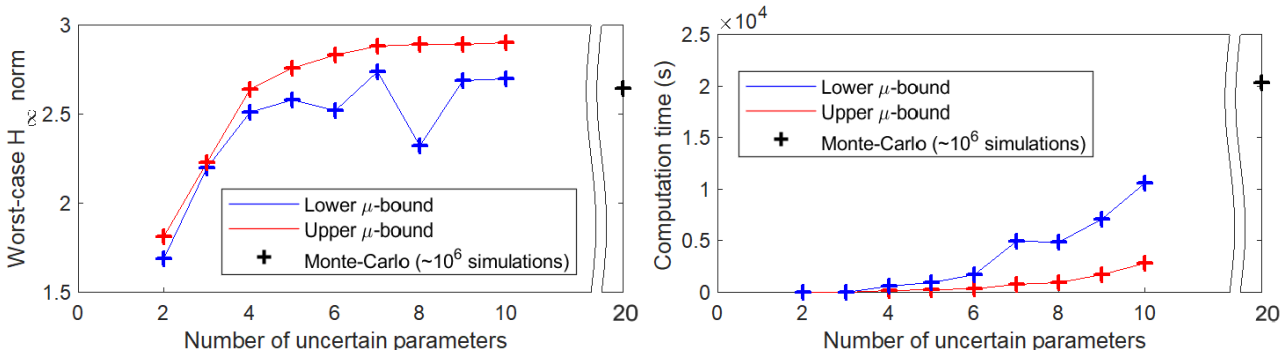
c) Computation times (Figure 9b): Red plot: the computation of the upper bounds is very fast when only the most influential parameters are kept, and increases significantly when least-influential parameters are added: e.g. from 338s with 6 parameters ($S_{Ti} \geq 0.03$) to 814s when adding the 7th parameter ($S_{Ti} = 0.003$). Blue plot: similar observation for the lower bounds: e.g. from 1704s with 6 parameters to 4976s with 7 parameters. Remark: since the procedure for the lower bounds is iterative (step i+1 is initialized with the results of step i), the cumulative times could also be considered.

In conclusion, the S_{Ti} provide a quantitative information to assess the trade-off between the accuracy of the μ-analysis and the computation time. The idea is that, when adding parameters who contribute the least to the variance, (i) the contributions to the worst-case performance should become increasingly small, and (ii) the worst-case configurations should become increasingly rare. With a reduced number of parameters, for example here keeping the first 6 parameters (that is, fixing the parameters whose total effects account for less than 0.3% of the unconditional variance), it is possible to quickly provide lower and upper bounds of the worst-case performance (of the reduced system). Regarding the lower bound, an iterative procedure may help the search, but the computation seems to become less reliable and slower when least influential parameters are added; the sensitivity analysis can provide some insight to make a choice on the parameters to be included. Nonetheless, the comparison with Monte-Carlo shows that even a rare worst-case performance of the full system can be found with limited number of parameters, and in shorter time.



(a) Worst-case $\mathscr{H}_\infty$ norm **(b) Computation time**

Fig. 9 Comparison of μ-analysis for different number of parameters, and Monte-Carlo.

5 Conclusion

This paper investigated three global sensitivity analysis techniques to study the influence of uncertain parameters on the $\mathscr{H}_\infty$ norm of a satellite benchmark. While the linear regression was not adequate to capture interactions between parameters, and in particular the presence of several candidate worst-case peaks, the Morris and variance methods allowed identifying the most influential parameters. Both methods can be used in combination, the former being computationally cheap and the latter providing more quantitative results. Convergence was also verified numerically. Finally, it was shown how the results can support the μ-analysis to efficiently characterize the worst-case performance.

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