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A Simple Quaternion Estimator with Error Analysis

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ABSTRACT

A simple single-frame quaternion estimator is introduced where two vector observations are deterministically processed. This estimator is amenable to analytical expressions for the bias and the covariance matrix of the estimation error under white noise assumptions in the body and reference vectors. The statistics are developed in four dimensions both for the additive and multiplicative quaternion errors, before being reduced to three dimensions. The exceedingly simple structure of the estimator is instrumental in reaching analytical expressions and a thorough treatment of its singularities. Moreover, the impact of normalization of the quaternion estimate is studied in generic terms clarifying a misconception about the singular nature of its covariance. The estimator's error analysis is tested for statistical consistency and its accuracy is illustrated via extensive Monte Carlo simulations. Results show that the proposed estimator performs as well or better than a state-of-the-art similar quaternion estimator with about half the computation burden.

Keywords: Attitude quaternion, vector measurement, bias and covariance analysis, quaternion normalization

Nomenclature

$\mathbf{q}$ = true quaternion
$\mathbf{r}_i$ = reference vector ($i = 1, 2$)
$\mathbf{b}_i$ = body vector ($i = 1, 2$)
$\mathcal{B}$ = body coordinate frame
$\mathcal{R}$ = reference coordinate frame
$\widehat{\mathbf{q}}$ = normalized quaternion estimate
$\Delta\bar{\mathbf{q}}$ = additive unnormalized quaternion error
$\Delta\widehat{\mathbf{q}}$ = additive quaternion error
$\delta\widehat{\mathbf{q}}$ = multiplicative quaternion error
$P_{\Delta\widehat{\mathbf{q}}}$ = covariance matrix $\Delta\widehat{\mathbf{q}}$
$P_{\delta\widehat{\mathbf{q}}}$ = covariance matrix $\delta\widehat{\mathbf{q}}$

1 Introduction

Attitude determination is critical to many aerospace missions and has several decades of history. The quaternion of rotation [1, p. 758], a singularity-free minimal attitude representation, is known to present excellent numerical and analytical properties and has become very popular for designing attitude estimators. An appealing advantage of single-frame attitude estimators is the lack of sensitivity to initial conditions since they provide global rather than incremental estimates. Rooted in the Wahba problem [2], the quaternion batch algorithm known as the q-method, early reported in [3], has given rise to sophisticated versions aiming at providing closed-form solutions and reducing the computational burden, e.g. [4–6] to cite a few. An error analysis of the q-method is revisited and extended to non-unit and noisy reference vectors in [7]. Excellent surveys of algorithms and error analyses can be found in [8] and [9, Chap. 5]. The q-method and related algorithms involve the solution of symmetric eigenvalues problems in dimension four, with computation burdens that naturally increase with the number of observations. Furthermore, there is a growing class of very small satellites, a.k.a. nanosatellites or CubeSats, that accommodate two sensors only onboard, e.g. a magnetometer and a Sun sensor. Early works in this realm are still used nowadays. The TRIAD estimator uses exactly two vector measurements and devises a virtual third one via an orthogonalization process in order to estimate the attitude matrix [10]. A covariance analysis of TRIAD for a multiplicative error in terms of the Euler vector is presented in [4]. A generalized TRIAD algorithm is introduced in [11] via gain optimization. Reference [12] presents an optimal attitude matrix estimator from two vector measurements in the realm of the Wahba problem. In [4] a closed-form expression of the QUEST algorithm is developed for the case of two observations. In [13] the EULER 2 estimator utilizes the Rodrigues formula for reconstructing the Euler axis/angle parameters and the attitude matrix. It invokes optimization of Wahba's loss function in the case of noisy measurements. It exploits a coplanarity condition of the measured and predicted vectors towards the development of analytical formulas. Reference [14] introduces a quaternion parameterization given a single vector observation. The degree of freedom is an angle around the observation. Several methods are devised for analytical determination of the attitude quaternion given two observations. Interestingly, the quaternions were manipulated as classes of equivalence where the elements are collinear but not necessarily unit-norm. Motivated by the insight provided in [14], an optimization method on the degrees of freedom in the quaternion parametrizations related to two observations is devised in [15]. Very efficient algorithms emerged: an optimal one, more accurate and as fast as TRIAD, and a suboptimal one, faster and as accurate as TRIAD.

This work is concerned with the development and analysis of a very fast quaternion estimator from two vector observations. It thoroughly addresses the singularity cases via sequential rotations, as noted in [15], it is preferable to choose a single desirable rotation as early as possible in order to save computations.. A deterministic error analysis is performed that lends itself to analytical expressions for the biases and covariances of the quaternion multiplicative and additive errors. Hinging on insights from [16], the quaternion is sought as the unique solution to a set of orthogonality conditions that involve simple linear expressions of the measurements. This approach fundamentally differs from [14, 15] because the quaternion is not sought in the space of some basis, and from [5] because the orthogonality conditions do not require solving for the optimal Wahba's loss value. The simplicity of the estimator also enables a thorough deterministic and random error analysis. It sheds additional light on the various nonlinear effects in quaternion estimation, including normalization. The analysis is carried out in four dimensions both for multiplicative and additive errors and lends itself to second-order and fourth-order expressions for the biases and for the errors covariance matrices, respectively, in terms of the measurement noises. The consistency of the approximations is verified via extensive Monte-Carlo simulations for representative test cases. Different Monte Carlo simulations are run to evaluate the proposed estimator's performances in terms of accuracy and computation burden. The novel estimator performs as accurately and twice as fast as one of the fast algorithms.

This paper is organized as follows. Section 2 presents preliminary results that are used in Section 3 for estimating the attitude quaternion. Section 4 addresses the singular cases via sequential rotation. Section 5 is concerned with the error analysis. Section 6 presents numerical results, and conclusions are drawn in Section 7. Detailed developments and proofs appear in the Appendix.

2 Preliminaries

This section follows Ref. [16]. Let $\mathbf{b}$ and $\mathbf{r}$ denote the projections of an ideal noise-free vector measurement on a body coordinate frame and a reference coordinate frame, $\mathcal{B}$ and $\mathcal{R}$, respectively. The rotation quaternion from $\mathcal{R}$ to $\mathcal{B}$, denoted by $\mathbf{q}$, belongs to the null space of the following matrix:

$$H = \begin{bmatrix} -\,[\mathbf{s}\times] & \mathbf{d} \\ -\mathbf{d}^T & 0 \end{bmatrix} \tag{1}$$

where

$$\mathbf{s} = \frac{1}{2}\left(\mathbf{b} + \mathbf{r}\right) \tag{2}$$

$$\mathbf{d} = \frac{1}{2}\left(\mathbf{b} - \mathbf{r}\right) \tag{3}$$

and $[\mathbf{s}\times]$ denotes the cross-product matrix built from the 3×1 vector $\mathbf{s}$. The spectral decomposition of the matrix H features a kernel, $\mathrm{Ker}H$, that is generated by the orthonormal basis $\{\mathbf{q}_1, \mathbf{q}_2\}$ where

$$\mathbf{q}_1 = \begin{bmatrix} \mathbf{s} \\ 0 \end{bmatrix} \frac{1}{||\mathbf{s}||} \tag{4}$$

$$\mathbf{q}_2 = \begin{bmatrix} -\mathbf{s} \times \mathbf{d} \\ ||\mathbf{s}||^2 \end{bmatrix} \frac{1}{||\mathbf{s}||} \tag{5}$$

Define $||\,||$ as the norm of a vector. In addition, the orthogonal complement plane to $\mathrm{Ker}H$, $(\mathrm{Ker}H)^{\perp}$, is generated by the orthonormal basis $\{\mathbf{q}_3, \mathbf{q}_4\}$ where

$$\mathbf{q}_3 = \begin{bmatrix} \mathbf{d} \\ 0 \end{bmatrix} \frac{1}{||\mathbf{d}||} \tag{6}$$

$$\mathbf{q}_4 = \begin{bmatrix} \mathbf{s} \times \mathbf{d} \\ ||\mathbf{d}||^2 \end{bmatrix} \frac{1}{||\mathbf{d}||} \tag{7}$$

Both $\mathbf{q}_1$ and $\mathbf{q}_2$ are feasible candidates to represent the rotation from $\mathcal{R}$ to $\mathcal{B}$ since they both belong to the null space of H. The quaternion $\mathbf{q}_1$ is characterized by a rotation angle of 180 degrees while $\mathbf{q}_2$ features a minimum angle. One may seek the true quaternion as a linear combination of $\mathbf{q}_1$ and $\mathbf{q}_2$. In this work, however, we follow an orthogonal route.

3 Closed-form Quaternion Estimation using Two Vector Observations

Given two ideal non-collinear vector observations associated with the same attitude, i.e., two pairs of noise-free unit-norm column-vectors, $(\mathbf{b}_1, \mathbf{r}_1)$ and $(\mathbf{b}_2, \mathbf{r}_2)$, such that the angle between $\mathbf{r}_1$ and $\mathbf{r}_2$ equals the angle between $\mathbf{b}_1$ and $\mathbf{b}_2$, then the true quaternion $\mathbf{q}$ is expressed as follows:

$$\mathbf{q} = \begin{bmatrix} \mathbf{d}_1 \times \mathbf{d}_2 \\ \mathbf{s}_1^T \mathbf{d}_2 \end{bmatrix} \frac{1}{\sqrt{||\mathbf{d}_1 \times \mathbf{d}_2||^2 + |\mathbf{s}_1^T \mathbf{d}_2|^2}} \tag{8}$$

where $\mathbf{s}_1, \mathbf{s}_2, \mathbf{d}_1, \mathbf{d}_2$ are defined from (2),(3). The proof follows. Let $\mathbf{q}_{ij}$ denote the j^{th} basis element, $j = 1, 2, 3, 4$, constructed from the i^{th} vector measurement, $i = 1, 2$. As noted in Section 2, the quaternion $\mathbf{q}$ belongs to the orthogonal complements of the two planes generated by the pairs $(\mathbf{q}_{13}, \mathbf{q}_{14})$ and $(\mathbf{q}_{23}, \mathbf{q}_{24})$, respectively. The sought quaternion can thus be found along the intersection of these planes. Let $\mathbf{x}$ denote a feasible unnormalized quaternion with vector part $\mathbf{a}$ and scalar part α, then the orthogonality relationships yield

$$\mathbf{d}_1^T \mathbf{a} = 0 \tag{9}$$

$$\mathbf{d}_2^T \mathbf{a} = 0 \tag{10}$$

$$(\mathbf{s}_1 \times \mathbf{d}_1)^T \mathbf{a} + \|\mathbf{d}_1\|^2 \alpha = 0 \tag{11}$$

$$(\mathbf{s}_2 \times \mathbf{d}_2)^T \mathbf{a} + \|\mathbf{d}_2\|^2 \alpha = 0 \tag{12}$$

Eqs. (9)-(12) are linearly dependent otherwise $\mathbf{q}$ would be null. For simplicity, we will use Eqs. (9)-(11), only. Eqs. (9)-(10) clearly show that a feasible choice for $\mathbf{a}$ is:

$$\mathbf{a} = \mathbf{d}_1 \times \mathbf{d}_2 \tag{13}$$

so that, using Eq. (13) in Eq. (11), α is determined as follows:

$$\begin{aligned}
(\mathbf{s}_1 \times \mathbf{d}_1)^T (\mathbf{d}_1 \times \mathbf{d}_2) + \|\mathbf{d}_1\|^2 \alpha &= 0 \\
\mathbf{s}_1^T \left[\mathbf{d}_1 \times\right] \left[\mathbf{d}_1 \times\right] \mathbf{d}_2 + \|\mathbf{d}_1\|^2 \alpha &= 0 \\
(\mathbf{s}_1^T \mathbf{d}_1) \mathbf{d}_1^T \mathbf{d}_2 - \|\mathbf{d}_1\|^2 \mathbf{s}_1^T \mathbf{d}_2 + \|\mathbf{d}_1\|^2 \alpha &= 0 \\
\|\mathbf{d}_1\|^2 (\alpha - \mathbf{s}_1^T \mathbf{d}_2) &= 0 \\
\alpha &= \mathbf{s}_1^T \mathbf{d}_2
\end{aligned}$$

which, upon normalization of the quaternion, yields the sought result. Finally, notice that due to the easily verifiable identity:

$$\mathbf{s}_1^T \mathbf{d}_2 = -\mathbf{s}_2^T \mathbf{d}_1$$

an identical expression for $\mathbf{q}$ is readily obtained by using Eq. (12) instead of Eq.(11).

4 Singularity Analysis via Sequential Rotations

Singularity happens in the estimator (8) when there is division by zero. In practice this seldom occurs because the noise in the data often prevents perfect cancellations of the vector differences or their parallelism. Hence, the estimator's equations remain valid with noisy data. Yet, when the noises are very low, these equations may be badly conditioned. This difficulty can be circumvented by the known method of sequential rotations [4, 5], [9, p. 192]. Thanks to the geometry insights we can find a priori feasible sequences for each singular case. Compared with a trial and error procedure, this saves computations as noted in [12]. The general approach is explained as follows and the particular sequences are proposed next. Assuming that the true attitude from $\mathcal{R}$ to $\mathcal{B}$ yields one of the singular cases, then a new reference frame C is sought such that the rotation from C to $\mathcal{B}$ yields well-behaved estimation equations. Once the quaternion from C to $\mathcal{B}$ is estimated, the quaternion from $\mathcal{R}$ to $\mathcal{B}$ may be retrieved by a simple composition with the quaternion from $\mathcal{R}$ to C, see Fig. 1.

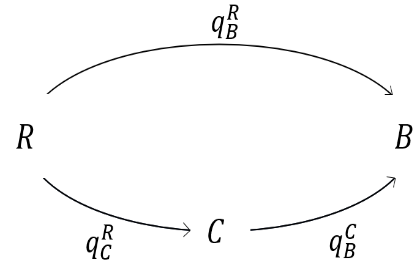

Fig. 1 Method of sequential rotations.

The steps are as follows: First of all, collect the VMs $(\mathbf{b}_1, \mathbf{r}_1)$ and $(\mathbf{b}_2, \mathbf{r}_2)$. Second, find a frame C, equivalently a quaternion $\mathbf{q}_C^{\mathcal{R}}$, such that the vector differences $\mathbf{d}_1^C, \mathbf{d}_2^C$ avoid any kind of singularity.n Thrid, Apply the estimator equations

$$\widehat{\mathbf{q}}_{\mathcal{B}}^C = \begin{bmatrix} \mathbf{d}_1^C \times \mathbf{d}_2^C \\ \mathbf{s}_1^C \cdot \mathbf{d}_2^C \end{bmatrix} \frac{1}{\sqrt{\|\mathbf{d}_1^C \times \mathbf{d}_2^C\|^2 + |\mathbf{s}_1^C \cdot \mathbf{d}_2^C|^2}} \tag{14}$$

where

$$\mathbf{s}_i^C = \frac{1}{2}(\mathbf{b}_i + \mathbf{r}_i^C), \quad i = 1, 2 \tag{15}$$

$$\mathbf{d}_i^C = \frac{1}{2}(\mathbf{b}_i - \mathbf{r}_i^C), \quad i = 1, 2 \tag{16}$$

$$\mathbf{r}_i^C = D_C^{\mathcal{R}}\, \mathbf{r}_i, \quad i = 1, 2 \tag{17}$$

where $D_C^{\mathcal{R}}$ denotes the DCM from $\mathcal{R}$ to C, and $\mathbf{r}_i^C$ denote the projections of the vector measurements on frame C.

Finally, calculate the sought quaternion:

$$\widehat{\mathbf{q}}_{\mathcal{B}}^{\mathcal{R}} = \widehat{\mathbf{q}}_C^{\mathcal{R}} * \widehat{\mathbf{q}}_{\mathcal{B}}^C \tag{18}$$

Typically, rotations of π radians about one of the standard axes are used for the transformation from $\mathcal{R}$ to C such that $\mathbf{q}_{\mathcal{B}}^{\mathcal{R}}$ is rewritten as follows:

$$\widehat{\mathbf{q}}_{\mathcal{B}_{R(\mathbf{x},\pi)}}^{\mathcal{R}} = [q_4, -q_3, q_2, -q_1]^T \tag{19}$$

$$\widehat{\mathbf{q}}_{\mathcal{B}_{R(\mathbf{y},\pi)}}^{\mathcal{R}} = [q_3, q_4, -q_1, -q_2]^T \tag{20}$$

$$\widehat{\mathbf{q}}_{\mathcal{B}_{R(\mathbf{z},\pi)}}^{\mathcal{R}} = [-q_2, q_1, q_4, -q_3]^T \tag{21}$$

where $R(\mathbf{x}, \pi)$ means rotate π along with the $\mathbf{x}$ axis and q_4 and $q_{1:3}$ are scalar and vector part of the quaternion $\widehat{\mathbf{q}}_{\mathcal{B}}^C$.

This results in simple permutations of the components of $\widehat{\mathbf{q}}_{\mathcal{B}}^C$ in Eq. (14) and of the components of $\mathbf{r}_i$ in Eq. (17), which saves computations. The development of simple validation criteria for each permutation is presented in A. Tab. 1 summarizes these validation conditions.

Table 1 Validation criteria for simple rotations

Singularity case	Validation criteria
$\mathbf{d}_1 = \mathbf{d}_2 = 0$	$R(\mathbf{x}, \pi)$ invalid IF $\mathbf{b}_1 \parallel$ x-axis OR $\mathbf{b}_2 \parallel$ x-axis OR $[0 \quad r_{1y} \quad r_{1z}] \parallel [0 \quad r_{2y} \quad r_{2z}]$ $R(\mathbf{y}, \pi)$ invalid IF $\mathbf{b}_1 \parallel$ y-axis OR $\mathbf{b}_2 \parallel$ y-axis OR $[r_{1x} \quad 0 \quad r_{1z}] \parallel [r_{2x} \quad 0 \quad r_{2z}]$ $R(\mathbf{z}, \pi)$ invalid IF $\mathbf{b}_1 \parallel$ z-axis OR $\mathbf{b}_2 \parallel$ z-axis OR $[r_{1x} \quad r_{1y} \quad 0] \parallel [r_{2x} \quad r_{2y} \quad 0]$
$\mathbf{d}_1 = 0$ and $\mathbf{d}_2 \neq 0$	$R(\mathbf{x}, \pi)$ invalid IF $\mathbf{b}_1 \parallel$ x-axis AND $(b_{2y} + r_{2y})r_{1z} = k(b_{2z} + r_{2z})r_{1y}$ $R(\mathbf{y}, \pi)$ invalid IF $\mathbf{b}_1 \parallel$ y-axis AND $(b_{2x} + r_{2x})r_{1z} = k(b_{2z} + r_{2z})r_{1x}$ $R(\mathbf{z}, \pi)$ invalid IF $\mathbf{b}_1 \parallel$ z-axis AND $(b_{2y} + r_{2y})r_{1x} = k(b_{2x} + r_{2x})r_{1y}$
$\mathbf{d}_1 \neq 0$ and $\mathbf{d}_2 = 0$	$R(\mathbf{x}, \pi)$ invalid IF $\mathbf{b}_2 \parallel$ x-axis AND $(b_{1y} + r_{1y})r_{2z} = k(b_{1z} + r_{1z})r_{2y}$ $R(\mathbf{y}, \pi)$ invalid IF $\mathbf{b}_2 \parallel$ y-axis AND $(b_{1x} + r_{1x})r_{2z} = k(b_{1z} + r_{1z})r_{2x}$ $R(\mathbf{z}, \pi)$ invalid IF $\mathbf{b}_2 \parallel$ z-axis AND $(b_{1y} + r_{1y})r_{2x} = k(b_{1x} + r_{1x})r_{2y}$
$\mathbf{d}_1 \times \mathbf{d}_2 = 0$ and $\mathbf{d}_1 \neq 0, \mathbf{d}_2 \neq 0$	$R(\mathbf{x}, \pi)$ invalid IF $\mathbf{d}_1 + [0 \quad r_{1y} \quad r_{1z}] \parallel \mathbf{d}_2 + [0 \quad r_{2y} \quad r_{2z}]$ $R(\mathbf{y}, \pi)$ invalid IF $\mathbf{d}_1 + [r_{1x} \quad 0 \quad r_{1z}] \parallel \mathbf{d}_2 + [r_{2x} \quad 0 \quad r_{2z}]$ $R(\mathbf{z}, \pi)$ invalid IF $\mathbf{d}_1 + [r_{1x} \quad r_{1y} \quad 0] \parallel \mathbf{d}_2 + [r_{2x} \quad r_{2y} \quad 0]$

5 Error Analysis

This section is concerned with the analysis of errors in the proposed estimator. We first introduce basic notations and definitions before performing a deterministic analysis. Second-order approximations of the estimation errors are easily developed despite the nonlinearity of the estimator. Also, the results are obtained related to the normalization effect and its impact on the additive and multiplicative quaternion errors. Then we proceed with a random error analysis focusing on the classical assumptions of additive unbiased white noises in the body and reference vectors and uncorrelation among the two vector measurements. The estimator's simplicity and the use of unnormalized variables help in streamlining the nonlinear analysis and developing closed-form expressions for the estimation error biases and covariance matrices of the quaternion additive and multiplicative errors and of the classical three-dimensional Euler vector error. The novelty of the current developments resides in better approximations of the biases and covariances, as illustrated in the numerical tests presented in the next section.

5.1 Definitions and notations

Let $\mathbf{\Delta b}_i, \mathbf{\Delta r}_i$ denote additive errors in the body frame and reference frame vectors, respectively:

$$\mathbf{b}_i = \mathbf{b}_i^t + \mathbf{\Delta b}_i \tag{22}$$

$$\mathbf{r}_i = \mathbf{r}_i^t + \mathbf{\Delta r}_i \tag{23}$$

where the superscript ^{t} denotes the true value of the underlying variable. Let $\bar{\mathbf{q}}$ and $\bar{\mathbf{q}}^t$ denote the unnormalized estimate and unnormalized true quaternion, respectively:

$$\bar{\mathbf{q}} = \begin{bmatrix} \mathbf{d}_1 \times \mathbf{d}_2 \\ \mathbf{s}_1 \cdot \mathbf{d}_2 \end{bmatrix} \tag{24}$$

$$\bar{\mathbf{q}}^t = \begin{bmatrix} \mathbf{d}_1^t \times \mathbf{d}_2^t \\ \mathbf{s}_1^t \cdot \mathbf{d}_2^t \end{bmatrix} \tag{25}$$

The additive estimation error in the unnormalized quaternion estimate, or simply the "unnormalized additive error", is defined as follows:

$$\Delta\bar{\mathbf{q}} = \bar{\mathbf{q}}^t - \bar{\mathbf{q}} \tag{26}$$

It will be handy, in the following developments, to use a scaled expression of the unnormalized additive error. Let $\check{\mathbf{q}}$ denote the following scaled estimate:

$$\check{\mathbf{q}} = \frac{\bar{\mathbf{q}}}{|\bar{\mathbf{q}}^t|} \tag{27}$$

where $|\bar{\mathbf{q}}^t|$ denotes the Euclidean norm of $\bar{\mathbf{q}}^t$, then the "scaled unnormalized error" is defined as follows:

$$\Delta\check{\mathbf{q}} = \frac{\Delta\bar{\mathbf{q}}}{|\bar{\mathbf{q}}^t|} \tag{28}$$

Furthermore, we define the "unnormalized multiplicative error" as follows:

$$\delta\bar{\mathbf{q}} = (\bar{\mathbf{q}}^t)^{-1} * \bar{\mathbf{q}} \tag{29}$$

where $(\bar{\mathbf{q}}^t)^{-1}$ denotes the quaternion inverse of $\bar{\mathbf{q}}^t$, and $*$ denotes the quaternion multiplication. Finally, let $\widehat{\mathbf{q}}$ and $\mathbf{q}$ denote the normalized estimated quaternion and the normalized true quaternion, which are defined as follows:

$$\widehat{\mathbf{q}} = \frac{\bar{\mathbf{q}}}{|\bar{\mathbf{q}}|} \tag{30}$$

$$\mathbf{q} = \frac{\bar{\mathbf{q}}^t}{|\bar{\mathbf{q}}^t|} \tag{31}$$

Then the additive and multiplicative errors in the normalized quaternion estimate, or simply, the "normalized additive and multiplicative errors", are defined as follows:

$$\Delta\widehat{\mathbf{q}} = \mathbf{q} - \widehat{\mathbf{q}} \tag{32}$$

$$\delta\widehat{\mathbf{q}} = (\mathbf{q})^{-1} * \widehat{\mathbf{q}} \tag{33}$$

5.2 Deterministic analysis

5.2.1 Exact formulas

In the following, an exact formula for the error $\Delta\bar{\mathbf{q}}$ is developed as a function of the measurement errors $\Delta\mathbf{b}$ and $\Delta\mathbf{r}$. Using Eq. (26) and the definitions of the vectors $\mathbf{s}$ and $\mathbf{d}$ yields the following expression for $\Delta\bar{\mathbf{q}}$:

$$\Delta\bar{\mathbf{q}} = \begin{bmatrix} [\mathbf{d}_2^t\times]\Delta\mathbf{d}_1 + [-\mathbf{d}_1^t\times]\Delta\mathbf{d}_2 - \Delta\mathbf{d}_1 \times \Delta\mathbf{d}_2 \\ -\mathbf{d}_2^t \cdot \Delta\mathbf{s}_1 - \mathbf{s}_1^t \cdot \Delta\mathbf{d}_2 - \Delta\mathbf{s}_1 \cdot \Delta\mathbf{d}_2 \end{bmatrix} \tag{34}$$

where the errors $\Delta\mathbf{d}_i$ and $\Delta\mathbf{s}_i$ are defined in Eqs. (2)- (3).

Next, we obtain exact expressions for the various errors as a function of $\mathbf{\Delta\bar{q}}$. Let ν denote the ratio between the norms of $\bar{\mathbf{q}}$ and $\bar{\mathbf{q}}^t$, i.e.

$$\nu = \frac{|\,\bar{\mathbf{q}}^t|}{|\,\bar{\mathbf{q}}|} \tag{35}$$

then the following identities are satisfied:

$$\boldsymbol{\delta}\bar{\mathbf{q}} = \mathbf{1_q} + M\mathbf{\Delta}\check{\mathbf{q}} \tag{36}$$

$$\nu = \left(1 - 2\mathbf{q}^T\mathbf{\Delta}\check{\mathbf{q}} + |\mathbf{\Delta}\check{\mathbf{q}}|^2\right)^{-1/2} \tag{37}$$

$$\widehat{\mathbf{\Delta q}} = \nu\mathbf{\Delta}\check{\mathbf{q}} + (1-\nu)\,\mathbf{q} \tag{38}$$

$$\widehat{\boldsymbol{\delta}\mathbf{q}} = \mathbf{1_q} + M\widehat{\mathbf{\Delta q}} \tag{39}$$

where $\mathbf{\Delta}\check{\mathbf{q}}$ is defined in Eq. (28), $\mathbf{1_q}$ and M are defined as follows:

$$\mathbf{\Delta}\check{\mathbf{q}} = \frac{\mathbf{\Delta\bar{q}}}{|\,\bar{\mathbf{q}}^t|} \tag{40}$$

$$\mathbf{1_q} = \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} \tag{41}$$

$$M = \begin{bmatrix} [\,\mathbf{e}\times] - q\,I_3 & \mathbf{e} \\ -\,\mathbf{e}^T & -q \end{bmatrix} \tag{42}$$

and $\mathbf{q}$ denotes the normalized true quaternion. Eqs. (34)-(42) are a set of exact formulas relating all errors to the underlying measurement errors. They are a useful preliminary to the development of approximations. The development of these formulas is provided in B.

5.2.2 *Second-order approximation formulas*

We aim here at developing approximate expressions for the errors that are accurate to second-order in $\mathbf{\Delta\bar{q}}$, or equivalently in $\mathbf{\Delta}\check{\mathbf{q}}$. Notice that the expression for the multiplicative error $\boldsymbol{\delta}\bar{\mathbf{q}}$, Eq.(36), is linear in $\mathbf{\Delta}\check{\mathbf{q}}$. Hence we first study ν then $\widehat{\mathbf{\Delta q}}$. The results are summarized in the following identities:

$$\nu = 1 + \mathbf{q}^T\mathbf{\Delta}\check{\mathbf{q}} + \frac{1}{2}\,\mathbf{\Delta}\check{\mathbf{q}}^T\left[3\mathbf{q}\mathbf{q}^T - I_4\right]\mathbf{\Delta}\check{\mathbf{q}} \tag{43}$$

$$\widehat{\mathbf{\Delta q}} = \left[\,I_4 - \mathbf{q}\mathbf{q}^T\right]\mathbf{\Delta}\check{\mathbf{q}} + \left[\mathbf{\Delta}\check{\mathbf{q}}\mathbf{\Delta}\check{\mathbf{q}}^T + \frac{1}{2}\,\mathbf{\Delta}\check{\mathbf{q}}^T\left(I_4 - 3\mathbf{q}\mathbf{q}^T\right)\mathbf{\Delta}\check{\mathbf{q}}\,I_4\right]\mathbf{q} \tag{44}$$

The accuracy of the above formulas depends on the amplitude of the higher-order terms. Developing the approximate expression for the multiplicative error $\widehat{\boldsymbol{\delta}\mathbf{q}}$ is straightforward, since $\widehat{\boldsymbol{\delta}\mathbf{q}}$ is linear in $\widehat{\mathbf{\Delta q}}$, see Eq.(39). The development of Eqs. (43)-(44) is provided in C. We notice in Eq. (44) that the first-order approximation to the normalized additive error is the projection of $\mathbf{\Delta}\check{\mathbf{q}}$ onto the orthogonal complement to the true quaternion. This is a well-known effect of quaternion normalization that creates an error lying close to the plane tangent to the unit sphere. The second-order factor however breaks this property by adding two terms: the first one is along the vector $\mathbf{\Delta}\check{\mathbf{q}}$ and the second along the true quaternion. Their amplitudes depend on the relative geometry of the scaled vector $\mathbf{\Delta\bar{q}}$ with $\mathbf{q}$, see a planar illustration in Fig. 2. To conclude, the current results provide handy formulas for studying the estimation errors as a function of $\mathbf{\Delta\bar{q}}$. The dependence upon the original errors in the body and reference vectors is readily obtained thanks to the exact Eqs. (34).

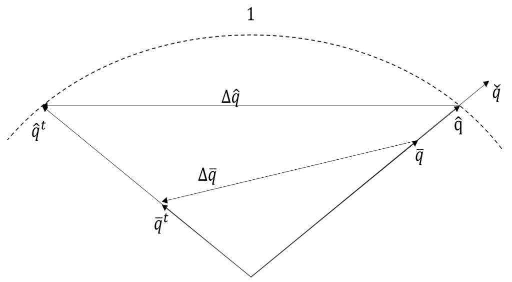

Fig. 2 Estimates and errors.

5.3 Random analysis

The deterministic error analysis easily lends itself to a random analysis under fairly general assumptions on the underlying vector measurement errors. In particular, we will provide simple expressions for the biases of the various errors that are accurate to the second order in $\mathbf{\Delta b}$ and $\mathbf{\Delta r}$. Subsequently, expressions for the covariance matrices of the various errors will be provided that are accurate to the fourth order in $\mathbf{\Delta b}$ and $\mathbf{\Delta r}$[17].

5.3.1 Biases

Assume that the errors $\mathbf{\Delta b}_1$, $\mathbf{\Delta b}_2$, $\mathbf{\Delta r}_1$, and $\mathbf{\Delta r}_2$ are unbiased and mutually uncorrelated random vectors. The unbiasedness holds when the vector measurements are not normalized or when their standard deviations σ are very small since their biases are of the order σ^2 [9, p. 204]. This is a common assumption given that biases in sensors are typically evaluated by calibration and can be compensated. Then the following identities hold:

$$E\{\mathbf{\Delta\bar{q}}\} = \mathbf{0} \tag{45}$$

$$E\{\mathbf{\Delta\check{q}}\} = \mathbf{0} \tag{46}$$

$$E\{\delta\mathbf{\bar{q}}\} = \mathbf{1_q} \tag{47}$$

$$E\{\nu\} = 1 - \frac{1}{2}\,\mathrm{tr}(QP_{\mathbf{\Delta\check{q}}}) \tag{48}$$

$$E\{\mathbf{\Delta\widehat{q}}\} = \left[P_{\mathbf{\Delta\check{q}}} + \frac{1}{2}\,\mathrm{tr}(QP_{\mathbf{\Delta\check{q}}})\,I_4\right]\mathbf{q} \tag{49}$$

$$E\{\delta\mathbf{\widehat{q}}\} = \mathbf{1_q} + M\left[P_{\mathbf{\Delta\check{q}}} + \frac{1}{2}\,\mathrm{tr}(QP_{\mathbf{\Delta\check{q}}})\,I_4\right]\mathbf{q} \tag{50}$$

where

$$Q = I_4 - 3\mathbf{q}\mathbf{q}^T \tag{51}$$

and $P_{\mathbf{\Delta\check{q}}}$ denotes the covariance matrix of $\mathbf{\Delta\check{q}}$, i.e. $P_{\mathbf{\Delta\check{q}}} = E\{\mathbf{\Delta\check{q}}\mathbf{\Delta\check{q}}^T\}$. The proofs of Eqs. (45)-(50) are provided in D. These results shed light on the impact of quaternion normalization. As seen from Eqs. (45)-(47), the unnormalized errors are unbiased. On the other hand, the normalized estimation errors, $\mathbf{\Delta\widehat{q}}$ and $\delta\mathbf{\widehat{q}}$, are biased, and their biases are functions of the covariance matrix of $P_{\mathbf{\Delta\check{q}}}$. A closed-form expression for the latter will be obtained in the next section. Furthermore, Eq. (49) shows that the bias includes a term along the true quaternion. This fact is similar to previous findings on unit vector measurement [9, sec. 5.5.2], whose bias lies opposite the true vector direction. Yet it is here unclear whether the bias points inward or outward to the unit sphere since the matrix Q is indefinite. Furthermore, the contribution of the term $P_{\Delta\check{\mathbf{q}}}\mathbf{q}$ to the bias is not necessarily along $\mathbf{q}$ although it is generally close to it. To conclude, the proposed estimator is biased, and closed-form expressions for the biases of the additive and multiplicative errors are developed. This paves the way for the development of the error covariance matrices.

5.3.2 The covariance matrix $P_{\Delta\bar{\mathbf{q}}}$

Since the equations for the unnormalized quaternion estimate $\bar{\mathbf{q}}$ are fairly simple, the derivation of the covariance matrix of $\Delta\bar{\mathbf{q}}$ is straightforward. Assume that the errors $\Delta\mathbf{b}_1$, $\Delta\mathbf{b}_2$, $\Delta\mathbf{r}_1$, and $\Delta\mathbf{r}_2$ have the same covariance matrix:

$$\mathrm{cov}\{\Delta\mathbf{b}_i\} = \sigma^2 I_3 \tag{52}$$

$$\mathrm{cov}\{\Delta\mathbf{r}_i\} = \sigma^2 I_3 \tag{53}$$

then the covariance matrix $P_{\Delta\bar{\mathbf{q}}}$ is expressed as follows:

$$P_{\Delta\bar{\mathbf{q}}} = \frac{\sigma^2}{2}\begin{bmatrix} \sum_{j=1}^{2} |\mathbf{d}_j^t|^2 I_3 - \mathbf{d}_j^t(\mathbf{d}_j^t)^T & \mathbf{d}_1^t \times \mathbf{s}_1^t \\ (\mathbf{d}_1^t \times \mathbf{s}_1^t)^T & |\mathbf{d}_2^t|^2 + |\mathbf{s}_1^t|^2 \end{bmatrix} \tag{54}$$

where $\mathbf{d}_1^t$, $\mathbf{d}_2^t$, $\mathbf{s}_1^t$ are defined in Eqs. (2)-(3). The impact of the noise level enters through the parameter σ^2 and the influence of the relative geometry of the vector measurements is expressed in the matrix. The derivation of Eq.(54) is provided in E along with the general case, where the $\Delta\mathbf{b}$ and $\Delta\mathbf{r}$ vectors are not necessarily uncorrelated and their covariance matrices may differ.

5.3.3 The other covariance matrices

Let $P_{\Delta\bar{\mathbf{q}}}$, $P_{\delta\bar{\mathbf{q}}}$, $P_{\Delta\widehat{\mathbf{q}}}$, $P_{\delta\widehat{\mathbf{q}}}$ denote the covariance matrices of $\Delta\bar{\mathbf{q}}$, $\delta\bar{\mathbf{q}}$, $\Delta\widehat{\mathbf{q}}$, and $\delta\widehat{\mathbf{q}}$, respectively. We present results on these covariance matrices for a generic $P_{\Delta\bar{\mathbf{q}}}$, which are summarized next:

$$P_{\Delta\check{\mathbf{q}}} = \frac{1}{|\bar{\mathbf{q}}^t|^2} P_{\Delta\bar{\mathbf{q}}} \tag{55}$$

$$P_{\delta\bar{\mathbf{q}}} = M P_{\Delta\check{\mathbf{q}}} M^T \tag{56}$$

$$P_{\Delta\widehat{\mathbf{q}}} = (I_4 - \mathbf{q}\mathbf{q}^T) P_{\Delta\check{\mathbf{q}}} (I_4 - \mathbf{q}\mathbf{q}^T)^T + N\mathbf{q}\mathbf{q}^T N^T \tag{57}$$

$$P_{\delta\widehat{\mathbf{q}}} = M P_{\Delta\widehat{\mathbf{q}}} M^T \tag{58}$$

where

$$N = P_{\Delta\check{\mathbf{q}}} + \frac{1}{2}\,\mathrm{tr}(Q P_{\Delta\check{\mathbf{q}}})\, I_4 \tag{59}$$

The proofs are provided in F. We notice that the covariance matrix $P_{\Delta\widehat{\mathbf{q}}}$ includes two terms. The first one is a similarity transformation of the matrix $P_{\Delta\check{\mathbf{q}}}$ where $(I_4 - \mathbf{q}\mathbf{q}^T)$ is an orthogonal projection matrix. That covariance is associated with the first-order (in $\Delta\check{\mathbf{q}}$) approximation of the estimation error. Since it lies on the tangential plane to the unit sphere, its components are linearly dependent and the associated covariance matrix is singular. The approximation to the second-order includes a term that lies out of the tangential plane (see Eq.(44)) and prevents, in general, the covariance matrix of $\Delta\widehat{\mathbf{q}}$ from being singular. The latter is true for $\delta\widehat{\mathbf{q}}$ since $P_{\delta\widehat{\mathbf{q}}}$ is obtained from $P_{\Delta\widehat{\mathbf{q}}}$ via a similarity transformation where M is an orthogonal matrix. These comments shed some light on a misconception in the field of quaternion estimation according to which the covariance matrix of the four-dimensional estimation error ought to be singular. Furthermore, these conclusions are general since our argument relies on a generic $P_{\Delta\bar{\mathbf{q}}}$. Indeed the above analysis focuses on the impact of the normalization. Finally, the expression for the Euler vector estimation error, $\boldsymbol{\delta\theta}$, can be derived using well-known approximations related to the multiplicative error $\widehat{\boldsymbol{\delta q}}$ [9, p. 201]:

$$\widehat{\boldsymbol{\delta q}} = \begin{bmatrix} \sin(\frac{\delta\theta}{2})\boldsymbol{\delta u} \\ \cos(\frac{\delta\theta}{2}) \end{bmatrix} \tag{60}$$

$$\simeq \begin{bmatrix} \frac{1}{2}\,\boldsymbol{\delta\theta} \\ 1 \end{bmatrix} \tag{61}$$

Therefore the Euler vector error, its bias, and its covariance matrix are approximated as follows:

$$\boldsymbol{\delta\theta} = 2\,\delta\hat{\mathbf{e}} \tag{62}$$

$$E\{\boldsymbol{\delta\theta}\} = 2\,E\{\delta\hat{\mathbf{e}}\} \tag{63}$$

$$\mathrm{cov}\{\boldsymbol{\delta\theta}\} = 4\ \mathrm{cov}\{\delta\hat{\mathbf{e}}\} \tag{64}$$

where $E\{\delta\hat{\mathbf{e}}\}$ and $\mathrm{cov}\{\delta\hat{\mathbf{e}}\}$ are readily extracted from $E\{\widehat{\boldsymbol{\delta q}}\}$ Eq.(50) and $P_{\widehat{\boldsymbol{\delta q}}}$ Eq.(58), respectively.

6 Test Results

In this section, we first verify the consistency of the proposed approximation statistics and then evaluate the estimator's performances via extensive Monte Carlo simulations.

6.1 Statistical consistency

The statistics formulas are verified in the particular case where the attitude is provided by the quaternion and the vector measurements in the reference frame are expressed as follows:

$$\mathbf{q} = \begin{bmatrix} 0 & 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

$$\mathbf{r}_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$\mathbf{r}_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

The additive noises $\boldsymbol{\Delta}\mathbf{b}$ and $\boldsymbol{\Delta}\mathbf{r}$ are simulated as zero-mean white Gaussian random variables with identical standard deviation $\sigma = 0.01$. A sample of 1,000,000 runs was created and the expected values were approximated using standard sample averaging. Tab. 2 summarizes the results. The columns 1 to 3 consist of the relative deviations between the predicted covariance matrices and their Monte Carlo values, expressed in percentage. Before or after the normalization, the relative deviations appear to be of the order of 0.2%. Also, the eigenvalues of $P_{\widehat{\boldsymbol{\delta q}}}$ and $P^{MC}_{\widehat{\boldsymbol{\delta q}}}$ are provided in the column 4. Again there is a very good agreement between the formulas and the actual values. Notice that the least eigenvalue is different from zero, which illustrates the positive definiteness of the covariance matrices. One can also notice that the least eigenvalue is very small compared to the other three, with ratios of about 10^{-4}, that is σ^2. This fact quantifies the bad conditioning of the covariance matrices for very small σ. Column 5 shows the deviations between the predicted biases and the Monte Carlo values, for both additive and multiplicative errors. They are both of order 10^{-5} which is one order of magnitude less than σ^2. Notice that $P_{\boldsymbol{\Delta}\bar{\mathbf{q}}}$ is used to calculate the biases and that the latter are needed in the calculations of the covariance matrices. Thus these results verify the statistical consistency of the proposed formulas for both the biases and the covariance matrices.

Table 2 Statistical consistency

$\frac{\lvert P^{MC}_{\Delta\bar{\mathbf{q}}} - P_{\Delta\bar{\mathbf{q}}}\rvert}{\lvert P^{MC}_{\Delta\bar{\mathbf{q}}}\rvert}$	$\frac{\lvert P^{MC}_{\Delta\widehat{\mathbf{q}}} - P_{\Delta\widehat{\mathbf{q}}}\rvert}{\lvert P^{MC}_{\Delta\widehat{\mathbf{q}}}\rvert}$	$\frac{\lvert P^{MC}_{\delta\widehat{\mathbf{q}}} - P_{\delta\widehat{\mathbf{q}}}\rvert}{\lvert P^{MC}_{\delta\widehat{\mathbf{q}}}\rvert}$	eigenvalue of $P^{MC}_{\Delta\widehat{\mathbf{q}}}, P_{\Delta\widehat{\mathbf{q}}} (\times 10^{-4})$	$[\overline{\Delta\hat{q}^{MC}} - E_{\Delta\hat{q}}, \overline{\delta\hat{q}^{MC}} - E_{\delta\hat{q}}](\times 10^{-4})$
0.16%	0.19%	0.19%	$\begin{bmatrix} 0.5062 & 0.5 \\ 0.4923 & 0.5 \\ 0.4787 & 0.5 \\ 0.000037 & 0.000037 \end{bmatrix}$	$\begin{bmatrix} -0.2357 & -0.2612 \\ 0.1336 & -0.0721 \\ -0.3446 & -0.4759 \\ -0.3285 & 0.0113 \end{bmatrix}$

We run similar Monte Carlo tests for various sample sizes, 1000, 10000, etc, in order to study the convergence rate of the different deviations. Tab. 3 summarizes the results. The second column to the left clearly depicts the asymptotic unbiasedness of the error $\Delta\bar{\mathbf{q}}$ (Euclidean norm) as N grows to infinity. The third column shows the decrease of the relative deviations in the covariance as N grows. The rate seems to be $1/\sqrt{N}$. The other columns depict the convergence of the deviations in the expectations and the covariances of the normalized errors to zero. In the latter too, the rate seems to be $1/\sqrt{N}$.

Table 3 Covariance and biases results

N	$\mid E^{MC}_{\Delta\bar{q}} \mid$	$\frac{\lvert P^{MC}_{\Delta\bar{\mathbf{q}}} - P_{\Delta\bar{\mathbf{q}}}\rvert}{\lvert P^{MC}_{\Delta\bar{\mathbf{q}}}\rvert}$	$\mid E^{MC}_{\Delta\hat{q}} - E_{\Delta\hat{q}} \mid, \mid E^{MC}_{\delta\hat{q}} - E_{\delta\hat{q}} \mid$	$\frac{\lvert P^{MC}_{\Delta\widehat{\mathbf{q}}} - P_{\Delta\widehat{\mathbf{q}}}\rvert}{\lvert P^{MC}_{\Delta\widehat{\mathbf{q}}}\rvert}, \frac{\lvert P^{MC}_{\delta\widehat{\mathbf{q}}} - P_{\delta\widehat{\mathbf{q}}}\rvert}{\lvert P^{MC}_{\delta\widehat{\mathbf{q}}}\rvert}$
N=1,000	3×10^{-4}	6%	2×10^{-4}	9%
N=10,000	1×10^{-4}	2.3%	2×10^{-4}	2.5%
N=100,000	3×10^{-5}	0.56%	3×10^{-5}	0.57%
N=1,000,000	4×10^{-6}	0.16%	4×10^{-6}	0.19%
N=10,000,000	4×10^{-6}	0.03%	5×10^{-6}	0.07%

6.2 Performance evaluation: accuracy and computation burden

To demonstrate the accuracy of our estimator, we simulate 100,000 cases using random attitude quaternions. The four quaternion components were sampled from a Gaussian distribution with zero mean and identity covariance, and then the sampled quaternion was normalized. For each attitude we generate two random reference vectors, $\mathbf{r}_1$ and $\mathbf{r}_2$, independently and uniformly distributed on the unit sphere. We use the attitude quaternion to map the reference vectors to the body frame. The vectors in reference and body frame are both corrupted by Gaussian random noise with a specified standard deviation of $1'$ i.e. about 0.016 degree. The estimator was applied at each run and produced an estimate $\widehat{\mathbf{q}}$. The accuracy measure was chosen to be the angular error $\delta\theta$ that is calculated as follows:

$$\delta\theta = 2 \arccos (\widehat{\mathbf{q}}^T \mathbf{q})$$

The performance of the estimator is depicted in Fig. 3 via the cumulative distribution function (CDF) of the estimation error. For the sake of comparison, we also plot the CDF of two state-of-the-art algorithms introduced in [15], named “Suboptimal Estimator” (here, Method 3) and “Optimal Estimator” (here Method 2). Both algorithms are very simple and their equations are re-written here for the sake of convenience (see Table 4). It appears that Method 2 outperforms Methods 1 and 3. On the other hand, our estimator outperforms Method 3 for errors greater than 0.16 degrees, which approximately corresponds to a probability upper bound of 0.93.

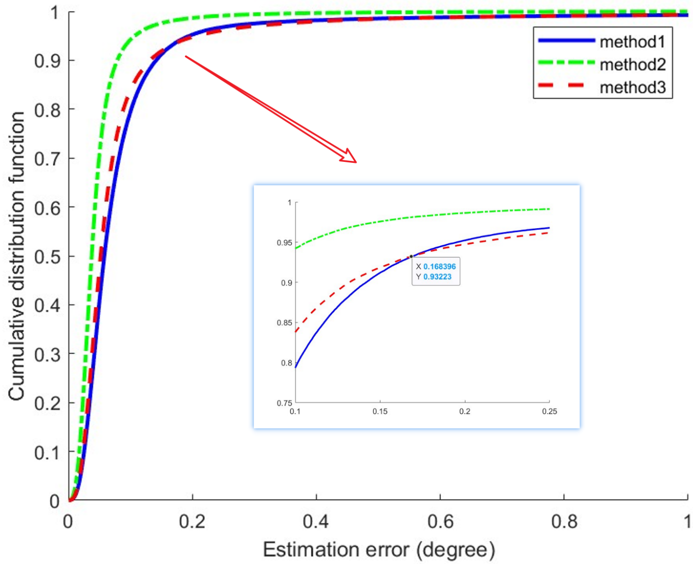

Fig. 3 Cumulative distribution function of $\delta\theta$

One way of measuring the computation burden is to count the number of floating point operations, or FLOP, in the estimators' main cycles [18]. Table 4 shows the number of FLOP for each of the three estimators. As shown in Table 4, Method 1 only requires 59 FLOP, versus 140 and 208 for the Method 3 and Method 2, respectively.

Table 4 Computation burden test

Case	Estimator	FLOPs
method 1	$\mathbf{s}_1 = \frac{1}{2}(\mathbf{b}_1 + \mathbf{r}_1)$ $\mathbf{d}_1 = \frac{1}{2}(\mathbf{b}_1 - \mathbf{r}_1); \mathbf{d}_2 = \frac{1}{2}(\mathbf{b}_2 - \mathbf{r}_2)$ $q = \begin{bmatrix} \mathbf{d}_1 \times \mathbf{d}_2 \\ \mathbf{s}_1^T \mathbf{d}_2 \end{bmatrix} \frac{1}{\sqrt{\lVert \mathbf{d}_1 \times \mathbf{d}_2 \rVert^2 + \lvert \mathbf{s}_1^T \mathbf{d}_2 \rvert^2}}$	59
method 3	$\mu = (1 + \mathbf{b}_1 \cdot \mathbf{r}_1) [(\mathbf{b}_1 \times \mathbf{b}_2) \cdot (\mathbf{r}_1 \times \mathbf{r}_2)] - [\mathbf{b}_1 \cdot (\mathbf{r}_1 \times \mathbf{r}_2)] [\mathbf{r}_1 \cdot (\mathbf{b}_1 \times \mathbf{b}_2)]$ $\nu = (\mathbf{b}_1 + \mathbf{r}_1) \cdot [(\mathbf{b}_1 \times \mathbf{b}_2) \times (\mathbf{r}_1 \times \mathbf{r}_2)]$ $\rho = \sqrt{\mu^2 + \nu^2}$ $q_{\text{TRIAD}} = \frac{1}{2\sqrt{\rho(\rho+\mu)(1 + \mathbf{b}_1 \cdot \mathbf{r}_1)}} \begin{bmatrix} (\rho+\mu)(\mathbf{b}_1 \times \mathbf{r}_1) + \nu(\mathbf{b}_1 + \mathbf{r}_i) \\ (\rho+\mu)(1 + \mathbf{b}_1 \cdot \mathbf{r}_1) \end{bmatrix}$ for $\mu \geq 0$ $q_{\text{TRIAD}} = \frac{1}{2\sqrt{\rho(\rho-\mu)(1 + \mathbf{b}_1 \cdot \mathbf{r}_1)}} \begin{bmatrix} \nu(\mathbf{b}_i \times \mathbf{r}_1) + (\rho-\mu)(\mathbf{b}_1 + \mathbf{r}_1) \\ \nu(1 + \mathbf{b}_1 \cdot \mathbf{r}_1) \end{bmatrix}$ for $\mu \leq 0$	140
method 2	$\mathbf{r}_3 = (\mathbf{r}_1 \times \mathbf{r}_2) / \lvert \mathbf{r}_1 \times \mathbf{r}_2 \rvert ; \mathbf{b}_3 = (\mathbf{b}_1 \times \mathbf{b}_2) / \lvert \mathbf{b}_1 \times \mathbf{b}_2 \rvert$ $\alpha = (1 + \mathbf{b}_3 \cdot \mathbf{r}_3)(a_1 \mathbf{b}_1 \cdot \mathbf{r}_1 + a_2 \mathbf{b}_2 \cdot \mathbf{r}_2) + (\mathbf{b}_3 \times \mathbf{r}_3) \cdot (a_1 \mathbf{b}_1 \times \mathbf{r}_1 + a_2 \mathbf{b}_2 \times \mathbf{r}_2)$ $\beta = (\mathbf{b}_3 + \mathbf{r}_3) \cdot (a_1 \mathbf{b}_1 \times \mathbf{r}_1 + a_2 \mathbf{b}_2 \times \mathbf{r}_2)$ $\gamma = \sqrt{\alpha^2 + \beta^2}$ $q_{\text{opt}} = \frac{1}{2\sqrt{\gamma(\gamma+\alpha)(1 + \mathbf{b}_3 \cdot \mathbf{r}_3)}} \begin{bmatrix} (\gamma+\alpha)(\mathbf{b}_3 \times \mathbf{r}_3) + \beta(\mathbf{b}_3 + \mathbf{r}_3) \\ (\gamma+\alpha)(1 + \mathbf{b}_3 \cdot \mathbf{r}_3) \end{bmatrix}$ for $\alpha \geq 0$ $q_{\text{opt}} = \frac{1}{2\sqrt{\gamma(\gamma-\alpha)(1 + \mathbf{b}_3 \cdot \mathbf{r}_3)}} \begin{bmatrix} \beta(\mathbf{b}_3 \times \mathbf{r}_3) + (\gamma-\alpha)(\mathbf{b}_3 + \mathbf{r}_3) \\ \beta(1 + \mathbf{b}_3 \cdot \mathbf{r}_3) \end{bmatrix}$ for $\alpha \leq 0$	208

For completeness, we checked the execution time of the three subroutines. The experiment was conducted with MATLAB 2022b running on a PC WIN64, featuring an Intel 12th Gen Core i5-12500H processor operating at 2.50 GHz. Table 5 summarizes the results.

Table 5 Computation running time test

Case	Average run-time [μsec]	Worst run-time [$msec$]	Ratio
method 1	0.8	0.1	-
method 3 [15]	1.5	0.2	1.8
method 2 [15]	2.1	0.2	2.6
q-method [9, sec. 5.3.1]	9.9	0.3	12.5
ESOQ [5], [9, sec. 5.3.5]	11.3	0.4	14.2
QUEST [4], [9, sec. 5.3.2]	12.9	0.7	16.2

Notice that the subroutines used here to solve for the quaternion via the q-method or ESOQ are built-in MATLAB codes. To conclude, the proposed estimator is typically as accurate as the suboptimal estimator of [15] and twice as fast.

7 Conclusion

The proposed algorithm is exceedingly simple. This dramatically simplifies the computations and it enables a systematic error analysis. In particular one can express the biases and the covariances of the additive errors, which is important for performance evaluations. The statistics were verified for consistency and the performances were compared with state-of-the-art algorithms. The accuracy of the proposed estimator compares well with an optimal similar quaternion estimator and is sometimes better than that of a related suboptimal version. An additional outcome of this study consists of the general analysis of the additive quaternion error: its covariance is in principle not singular despite normalization. The proposed approximations shed light on the conditions under which the matrix may become badly conditioned.

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Appendix

A Validation criteria for simple sequences of rotations

The validation criteria are developed for each singularity case.

Case A: $\mathbf{d}_1 = \mathbf{d}_2 = 0$

Assume a rotation of π radians about the x-axis:

$$\mathbf{b}_1 = \mathbf{r}_1 = [r_{1x}, r_{1y}, r_{1z}]^T \tag{A1}$$

$$\mathbf{b}_2 = \mathbf{r}_2 = [r_{2x}, r_{2y}, r_{2z}]^T \tag{A2}$$

$$\mathbf{r}_1^C = [r_{1x}, -r_{1y}, -r_{1z}]^T \tag{A3}$$

$$\mathbf{r}_2^C = [r_{2x}, -r_{2y}, -r_{2z}]^T \tag{A4}$$

$$\mathbf{d}_1^C = [0, r_{1y}, r_{1z}]^T \tag{A5}$$

$$\mathbf{d}_2^C = [0, r_{2y}, r_{2z}]^T \tag{A6}$$

If $\mathbf{b}_1 \parallel$ x-axis or $\mathbf{b}_2 \parallel$ x-axis or $[0, r_{1y}, r_{1z}]^T \parallel [0, r_{2y}, r_{2z}]^T$, then the vector differences $\mathbf{d}_1^C, \mathbf{d}_2^C$ are in one of the singular cases. Similar conclusions are readily obtained for rotations around the other two axes by π radians.

Case B: $\mathbf{d}_1 = 0$ and $\mathbf{d}_2 \neq 0$

Assume a rotation of π radians about the x-axis:

$$\mathbf{b}_1 = \mathbf{r}_1 = [r_{1x}, r_{1y}, r_{1z}]^T \tag{A7}$$

$$\mathbf{r}_2 = [r_{2x}, r_{2y}, r_{2z}]^T \tag{A8}$$

$$\mathbf{b}_2 = [b_{2x}, b_{2y}, b_{2z}]^T \tag{A9}$$

$$\mathbf{r}_1^C = [r_{1x}, -r_{1y}, -r_{1z}]^T \tag{A10}$$

$$\mathbf{r}_2^C = [r_{2x}, -r_{2y}, -r_{2z}]^T \tag{A11}$$

$$\mathbf{d}_1^C = [0, r_{1y}, r_{1z}]^T \tag{A12}$$

$$\mathbf{d}_2^C = \frac{1}{2}[b_{2x} - r_{2x}, b_{2y} + r_{2y}, b_{2z} + r_{2z}]^T \tag{A13}$$

If $\mathbf{b}_1 \parallel$ x-axis and $(b_{2y} + r_{2y})r_{1z} = k(b_{2z} + r_{2z})r_{1y}$, where k is an arbitrary constant, then the vector differences $\mathbf{d}_1^C$ and $\mathbf{d}_2^C$ are in one of the singular cases. Similar conclusions are readily obtained for rotations around the other two axes by π radians.

Case C: $\mathbf{d}_1 \neq 0$ and $\mathbf{d}_2 = 0$

Assume a rotation of π radians about the x-axis:

$$\mathbf{r}_1 = [r_{1x}, r_{1y}, r_{1z}]^T \tag{A14}$$

$$\mathbf{b}_1 = [b_{1x}, b_{1y}, b_{1z}]^T \tag{A15}$$

$$\mathbf{b}_2 = \mathbf{r}_2 = [r_{2x}, r_{2y}, r_{2z}]^T \tag{A16}$$

$$\mathbf{r}_1^C = [r_{1x}, -r_{1y}, -r_{1z}]^T \tag{A17}$$

$$\mathbf{r}_2^C = [r_{2x}, -r_{2y}, -r_{2z}]^T \tag{A18}$$

$$\mathbf{d}_1^C = \frac{1}{2}[b_{1x} - r_{1x}, b_{1y} + r_{1y}, b_{1z} + r_{1z}]^T \tag{A19}$$

$$\mathbf{d}_2^C = [0, r_{2y}, r_{2z}]^T \tag{A20}$$

If $\mathbf{b}_2 \parallel$ x-axis and $(b_{1y}+r_{1y})r_{2z} = k(b_{1z}+r_{1z})r_{2y}$, where k is an arbitrary constant, then the vector differences $\mathbf{d}_1^C, \mathbf{d}_2^C$ are in one of the singular cases. Similar conclusions are readily obtained for rotations around the other two axes by π radians.

Case D: $\mathbf{d}_1 \times \mathbf{d}_2 = 0$ and $\mathbf{d}_1 \neq 0, \mathbf{d}_2 \neq 0$

Assume a rotation of π radians about the x-axis:

$$\mathbf{r}_1 = [r_{1x}, r_{1y}, r_{1z}]^T \tag{A21}$$
$$\mathbf{r}_2 = [r_{2x}, r_{2y}, r_{2z}]^T \tag{A22}$$
$$\mathbf{b}_1 = [b_{1x}, b_{1y}, b_{1z}]^T \tag{A23}$$
$$\mathbf{b}_2 = [b_{2x}, b_{2y}, b_{2z}]^T \tag{A24}$$
$$\mathbf{r}_1^C = [r_{1x}, -r_{1y}, -r_{1z}]^T \tag{A25}$$
$$\mathbf{r}_2^C = [r_{2x}, -r_{2y}, -r_{2z}]^T \tag{A26}$$
$$\mathbf{d}_1^C = \frac{1}{2}[b_{1x}-r_{1x}, b_{1y}+r_{1y}, b_{1z}+r_{1z}]^T \tag{A27}$$
$$\mathbf{d}_2^C = \frac{1}{2}[b_{2x}-r_{2x}, b_{2y}+r_{2y}, b_{2z}+r_{2z}]^T \tag{A28}$$

If $\mathbf{d}_1 + [0, r_{1y}, r_{1z}]^T \parallel \mathbf{d}_2 + [0, r_{2y}, r_{2z}]^T$, then the vector differences $\mathbf{d}_1^C, \mathbf{d}_2^C$ are in one of the singular cases. Similar conclusions are readily obtained for rotations around the other two axes by π radians.

B Proof of Eqs. (34)-(42)

The proof of Eq.(34) is straightforward and relies on perturbation of the expression for the unnormalized estimate $\bar{\mathbf{q}}^t$. The unnormalized multiplicative error $\boldsymbol{\delta}\bar{\mathbf{q}}$ is defined in Eq.(29) using the quaternion inverse, as follows:

$$\begin{aligned}
\delta\bar{\mathbf{q}} &= (\bar{\mathbf{q}}^t)^{-1} * \bar{\mathbf{q}} \\
&= (\bar{\mathbf{q}}^t)^{-1} * (\bar{\mathbf{q}}^t - \Delta\bar{\mathbf{q}}) \\
&= (\bar{\mathbf{q}}^t)^{-1} * \bar{\mathbf{q}}^t - (\bar{\mathbf{q}}^t)^{-1} * \Delta\bar{\mathbf{q}} \\
&= \mathbf{1}_\mathbf{q} - (\bar{\mathbf{q}}^t)^{-1} * \Delta\bar{\mathbf{q}} \\
&= \mathbf{1}_\mathbf{q} - |\bar{\mathbf{q}}^t|(\bar{\mathbf{q}}^t)^{-1} * \Delta\check{\mathbf{q}}
\end{aligned}$$

and Eq. (36) follows using Eq.(28), Eq.(31). A similar development is readily applied in order to obtain Eq.(39). Using the definition of $\boldsymbol{\delta}\widehat{\mathbf{q}}$ in Eq.(33) yields

$$\begin{aligned}
\delta\widehat{\mathbf{q}} &= (\mathbf{q})^{-1} * \widehat{\mathbf{q}} \\
&= (\mathbf{q})^{-1} * (\mathbf{q} - \Delta\widehat{\mathbf{q}}) \\
&= (\mathbf{q})^{-1} * \mathbf{q} - (\mathbf{q})^{-1} * \Delta\widehat{\mathbf{q}} \\
&= \mathbf{1}_\mathbf{q} - (\mathbf{q})^{-1} * \Delta\widehat{\mathbf{q}} \\
&= \mathbf{1}_\mathbf{q} - \mathbf{q}^{-1} * \Delta\widehat{\mathbf{q}}
\end{aligned}$$

and Eq.(39) follows. Next Eq.(37) is developed by considering the squared norm of $\bar{\mathbf{q}}$ first.

$$\begin{aligned}|\bar{\mathbf{q}}|^2 &= |\bar{\mathbf{q}}^t - \Delta\bar{\mathbf{q}}|^2 \\ &= |\bar{\mathbf{q}}^t|^2 - 2\,\bar{\mathbf{q}}^t \cdot \Delta\bar{\mathbf{q}} + |\Delta\bar{\mathbf{q}}|^2\end{aligned}$$

Dividing by $|\bar{\mathbf{q}}^t|^2$ yields

$$\begin{aligned}\frac{|\bar{\mathbf{q}}|^2}{|\bar{\mathbf{q}}^t|^2} &= 1 - 2\frac{\bar{\mathbf{q}}^t}{|\bar{\mathbf{q}}^t|} \cdot \frac{\Delta\bar{\mathbf{q}}}{|\bar{\mathbf{q}}^t|} + \left(\frac{|\Delta\bar{\mathbf{q}}|}{|\bar{\mathbf{q}}^t|}\right)^2 \\ &= 1 - 2\mathbf{q} \cdot \Delta\check{\mathbf{q}} + |\Delta\check{\mathbf{q}}|^2\end{aligned}$$

and Eq.(37) follows by the operations of square root and inversion. Finally, Eq.(38) is developed as follows.

$$\begin{aligned}\Delta\widehat{\mathbf{q}} &= \mathbf{q} - \widehat{\mathbf{q}} \\ &= \frac{\bar{\mathbf{q}}^t}{|\bar{\mathbf{q}}^t|} - \frac{\bar{\mathbf{q}}}{|\bar{\mathbf{q}}|} \\ &= \frac{\bar{\mathbf{q}}^t}{|\bar{\mathbf{q}}^t|} - \frac{\bar{\mathbf{q}}^t - \Delta\bar{\mathbf{q}}}{|\bar{\mathbf{q}}|} \\ &= \frac{\Delta\bar{\mathbf{q}}}{|\bar{\mathbf{q}}|} + \left(\frac{1}{|\bar{\mathbf{q}}^t|} - \frac{1}{|\bar{\mathbf{q}}|}\right)\bar{\mathbf{q}}^t \\ &= \frac{|\bar{\mathbf{q}}^t|}{|\bar{\mathbf{q}}|}\frac{\Delta\bar{\mathbf{q}}}{|\bar{\mathbf{q}}^t|} + \left(1 - \frac{|\bar{\mathbf{q}}^t|}{|\bar{\mathbf{q}}|}\right)\frac{\bar{\mathbf{q}}^t}{|\bar{\mathbf{q}}^t|}\end{aligned}$$

and Eq.(38) follows using the definitions of $\mathbf{q}$, $\Delta\check{\mathbf{q}}$, and ν.

C Proof of Eqs. (43)-(44)

From Eq. (37), the ratio ν is expressed as follows:

$$\nu = (1 + \epsilon)^{-\frac{1}{2}}$$

where

$$\epsilon = -2\mathbf{q}^T\Delta\check{\mathbf{q}} + |\Delta\check{\mathbf{q}}|^2$$

A power series expansion to the second order in ϵ yields

$$\begin{aligned}\nu &= 1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 \\ &= 1 + \mathbf{q}^T\Delta\check{\mathbf{q}} - \frac{1}{2}\Delta\check{\mathbf{q}}^T\Delta\check{\mathbf{q}} + \frac{3}{2}\Delta\check{\mathbf{q}}^T\mathbf{q}\mathbf{q}^T\Delta\check{\mathbf{q}}\end{aligned}$$

and Eq.(43) follows. Inserting Eq. (43) into Eq.(38), keeping the second order terms in $\Delta\check{\mathbf{q}}$ and rearranging yields Eq. (44).

D Proof of Eqs. (45)-(50)

We first notice that $\Delta\mathbf{s}_1$, $\Delta\mathbf{d}_1$ and $\Delta\mathbf{d}_2$ are linear transformations of $\Delta\mathbf{b}_1$, $\Delta\mathbf{b}_2$, $\Delta\mathbf{r}_1$, and $\Delta\mathbf{r}_2$, and are thus unbiased, and mutually uncorrelated. Then applying the expectation operator to Eq. (34), using the linearity property, the unbiasedness of $\Delta\mathbf{s}$ and $\Delta\mathbf{d}$ and their uncorrelation, yields the sought result, i.e., the unbiasedness of $\Delta\bar{\mathbf{q}}$ Eq.(45). Since $\Delta\check{\mathbf{q}}$ is obtained through the division of $\Delta\bar{\mathbf{q}}$ by $|\bar{\mathbf{q}}^t|$, it is unbiased as well and Eq.(46) follows. Eq. (47) follows from Eq.(46) and Eq.(36). Eq. (48) is obtained as follows:

$$
\begin{aligned}
E\{\nu\} &= E\{1 + \mathbf{q}^T\Delta\check{\mathbf{q}} + \frac{1}{2}\Delta\check{\mathbf{q}}^T\left(3\mathbf{q}\mathbf{q}^T - I_4\right)\Delta\check{\mathbf{q}}\} \\
&= 1 + \mathbf{q}^T E\{\Delta\check{\mathbf{q}}\} + \frac{1}{2}E\{\Delta\check{\mathbf{q}}^T\left(3\mathbf{q}\mathbf{q}^T - I_4\right)\Delta\check{\mathbf{q}}\} \\
&= 1 + \frac{1}{2}\,\mathrm{tr}\left[\left(3\mathbf{q}\mathbf{q}^T - I_4\right)E\{\Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T\}\right]
\end{aligned}
$$

Eq. (49) is obtained in a similar manner:

$$
\begin{aligned}
E\{\Delta\widehat{\mathbf{q}}\} &= E\{\left[I_4 - \mathbf{q}\mathbf{q}^T\right]\Delta\check{\mathbf{q}} + \left[\Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T + \frac{1}{2}\Delta\check{\mathbf{q}}^T\left(I_4 - 3\mathbf{q}\mathbf{q}^T\right)\Delta\check{\mathbf{q}}\,I_4\right]\mathbf{q}\} \\
&= \left[I_4 - \mathbf{q}\mathbf{q}^T\right]E\{\Delta\check{\mathbf{q}}\} + \left[E\{\Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T\} + \frac{1}{2}E\{\Delta\check{\mathbf{q}}^T\left(I_4 - 3\mathbf{q}\mathbf{q}^T\right)\Delta\check{\mathbf{q}}\}\,I_4\right]\mathbf{q} \\
&= \left\{P_{\Delta\check{\mathbf{q}}} + \frac{1}{2}\,\mathrm{tr}\left[\left(I_4 - 3\mathbf{q}\mathbf{q}^T\right)P_{\Delta\check{\mathbf{q}}}\right]I_4\right\}\mathbf{q}
\end{aligned}
$$

Finally using Eq.(49) in Eq.(39) yields Eq.(50).

E Derivation of the covariance matrix $P_{\Delta\bar{\mathbf{q}}}$

E1 Covariance matrices of $\Delta\mathbf{d}$ and $\Delta\mathbf{s}$

Let $P_{\Delta\mathbf{d}}$ and $P_{\Delta\mathbf{s}}$ denote the covariance matrices of $\Delta\mathbf{d}$ and $\Delta\mathbf{s}$, respectively. Recalling their expressions:

$$\Delta\mathbf{s} = \frac{1}{2}\left(\Delta\mathbf{b} + \Delta\mathbf{r}\right) \tag{E1}$$

$$\Delta\mathbf{d} = \frac{1}{2}\left(\Delta\mathbf{b} - \Delta\mathbf{r}\right) \tag{E2}$$

yields

$$P_{\Delta\mathbf{d}} = \frac{1}{4}\begin{bmatrix} I_3 & -I_3 \end{bmatrix}\begin{bmatrix} P_{\Delta\mathbf{b}} & P_{\Delta\mathbf{b}\Delta\mathbf{r}} \\ P^T_{\Delta\mathbf{b}\Delta\mathbf{r}} & P_{\Delta\mathbf{s}} \end{bmatrix}\begin{bmatrix} I_3 \\ -I_3 \end{bmatrix} \tag{E3}$$

$$P_{\Delta\mathbf{s}} = \frac{1}{4}\begin{bmatrix} I_3 & I_3 \end{bmatrix}\begin{bmatrix} P_{\Delta\mathbf{b}} & P_{\Delta\mathbf{b}\Delta\mathbf{r}} \\ P^T_{\Delta\mathbf{b}\Delta\mathbf{r}} & P_{\Delta\mathbf{s}} \end{bmatrix}\begin{bmatrix} I_3 \\ I_3 \end{bmatrix} \tag{E4}$$

where $P_{\Delta\mathbf{b}}$, $P_{\Delta\mathbf{r}}$, and $P_{\Delta\mathbf{b}\Delta\mathbf{r}}$ denote the covariance matrices of $\Delta\mathbf{b}$, $\Delta\mathbf{r}$, and the cross-covariance matrix, respectively. Cross-covariances $P_{\Delta\mathbf{d}_1\Delta\mathbf{s}_1}$, $P_{\Delta\mathbf{d}_2\Delta\mathbf{s}_1}$ $P_{\Delta\mathbf{d}_1\Delta\mathbf{d}_2}$ are easily derived by similar arguments.

E2 Covariance matrix $P_{\Delta\bar{\mathbf{q}}}$: the general case

Recalling the expression of $\Delta\bar{\mathbf{q}}$ from Eq.(34) and retaining the linear terms only yields:

$$\Delta\bar{\mathbf{q}} = \begin{bmatrix} 0_{3\times3} & E & F \\ G & 0_{1\times3} & H \end{bmatrix} \begin{bmatrix} \Delta\mathbf{s}_1 \\ \Delta\mathbf{d}_1 \\ \Delta\mathbf{d}_2 \end{bmatrix} \tag{E5}$$

where

$$E = [\mathbf{d}_2^t \times] \tag{E6}$$

$$F = [-\mathbf{d}_1^t \times] \tag{E7}$$

$$G = (-\mathbf{d}_2^t)^T \tag{E8}$$

$$H = (-\mathbf{s}_1^t)^T \tag{E9}$$

Hence the covariance matrix $P_{\Delta\bar{\mathbf{q}}}$ is expressed as follows

$$P_{\Delta\bar{\mathbf{q}}} = \begin{bmatrix} 0_{3\times3} & E & F \\ G & 0_{1\times3} & H \end{bmatrix} \begin{bmatrix} P_{\Delta\mathbf{s}_1} & P_{\Delta\mathbf{s}_1\Delta\mathbf{d}_1} & P_{\Delta\mathbf{s}_1\Delta\mathbf{d}_2} \\ P^T_{\Delta\mathbf{s}_1\Delta\mathbf{d}_1} & P_{\Delta\mathbf{d}_1} & P_{\Delta\mathbf{d}_1\Delta\mathbf{d}_2} \\ P^T_{\Delta\mathbf{s}_1\Delta\mathbf{d}_2} & P^T_{\Delta\mathbf{d}_1\Delta\mathbf{d}_2} & P_{\Delta\mathbf{d}_2} \end{bmatrix} \begin{bmatrix} 0_{3\times3} & G^T \\ E^T & 0_{3\times1} \\ F^T & H^T \end{bmatrix} \tag{E10}$$

E3 Covariance matrix $P_{\Delta\bar{\mathbf{q}}}$: the particular case

Under the assumptions of independence and identical covariances for the vector measurement errors, $P_{\Delta\bar{\mathbf{q}}}$ boils down to the following matrix:

$$\begin{aligned} P_{\Delta\bar{\mathbf{q}}} &= \frac{\sigma^2}{2} \begin{bmatrix} 0_{3\times3} & E & F \\ G & 0_{1\times3} & H \end{bmatrix} \begin{bmatrix} I_3 & I_3 & 0_{3\times3} \\ I_3 & I_3 & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} & I_3 \end{bmatrix} \begin{bmatrix} 0_{3\times3} & G^T \\ E^T & 0_{3\times1} \\ F^T & H^T \end{bmatrix} \\ &= \frac{\sigma^2}{2} \begin{bmatrix} EE^T + FF^T & FH^T \\ HF^T & GG^T + HH^T \end{bmatrix} \\ &= \frac{\sigma^2}{2} \begin{bmatrix} \sum_{j=1}^2 [\mathbf{d}_j^t \times][\mathbf{d}_j^t \times]^T & [-\mathbf{d}_1^t \times](-\mathbf{s}_1^t) \\ (-\mathbf{s}_1^t)^T [-\mathbf{d}_1^t \times]^T & \mathbf{d}_2^t \cdot \mathbf{d}_2^t + \mathbf{s}_1^t \cdot \mathbf{s}_1^t \end{bmatrix} \end{aligned}$$

and Eq.(54) follows.

F Proof of Eqs. (55)-(58)

Equations Eqs.(55)- (56) stem directly from the definition of $\Delta\check{\mathbf{q}}$, and from Eq.(36) where $\delta\bar{\mathbf{q}}$ is shown to be affine with respect to $\Delta\check{\mathbf{q}}$. For Eq.(57), let N_2 and N denote the following matrices:

$$N_2 = \Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T + \frac{1}{2}\,\mathrm{tr}(Q\Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T)\, I_4$$

$$N = P_{\Delta\check{\mathbf{q}}} + \frac{1}{2}\,\mathrm{tr}(QP_{\Delta\check{\mathbf{q}}})\, I_4$$

Then consider the deviation:

$$\boldsymbol{\Delta}\widehat{\mathbf{q}} - E\{\boldsymbol{\Delta}\widehat{\mathbf{q}}\} = (I_4 - \mathbf{q}\mathbf{q}^T)\boldsymbol{\Delta}\check{\mathbf{q}} + (N_2 - N)\mathbf{q}$$

Eq. (57) follows by applying the expectation operator while retaining only the second-order terms in $\boldsymbol{\Delta}\check{\mathbf{q}}$. Eq. (58) stems from the fact that $\boldsymbol{\delta}\widehat{\mathbf{q}}$ is affine in $\boldsymbol{\Delta}\widehat{\mathbf{q}}$.